UNIFORM ASYMPTOTICS FOR DISCRETE ORTHOGONAL POLYNOMIALS WITH RESPECT TO VARYING EXPONENTIAL WEIGHTS ON A REGULAR INFINITE LATTICE

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ABSTRACT. We consider the large N asymptotics of a system of discrete orthogonal polynomials on an infinite regular lattice of mesh $\frac{1}{N}$, with weight $e^{-NV(x)}$, where $V(x)$ is a real analytic function with sufficient growth at infinity. The proof is based on formulation of an interpolation problem for discrete orthogonal polynomials, which can be converted to a Riemann-Hilbert problem, and steepest descent analysis of this Riemann-Hilbert problem.

1. INTRODUCTION

For a given $N \in \mathbb{N}$, introduce the regular infinite lattice,

$$L_N = \left\{ x_{k,N} = \frac{k}{N}, \ k \in \mathbb{Z} \right\}. \tag{1.1}$$

We consider polynomials orthogonal on L_N with respect to the varying exponential weight

$$w_N(x) = e^{-NV(x)}, \tag{1.2}$$

where $V(x)$ is a real analytic function such that, for some $\varepsilon > 0$, V has analytic extension into the strip

$$|\operatorname{Im} z| < \varepsilon, \tag{1.3}$$

and satisfies the growth condition

$$\frac{\operatorname{Re} V(z)}{\log(|z|^2 + 1)} \to +\infty \quad \text{as } |z| \to \infty, \ |\operatorname{Im} z| < \varepsilon. \tag{1.4}$$

More specifically, we introduce the system of monic orthogonal polynomials,

$$P_n(x) = x^n + p_{n,n-1} x^{n-1} + \ldots + p_{n0}, \quad n = 0, 1, \ldots,$$

such that

$$\sum_{x \in L_N} P_m(x) P_n(x) w_N(x) = h_n \delta_{mn}, \tag{1.5}$$

for some normalizing coefficients h_n. Existence and uniqueness of this system of orthogonal polynomials is guaranteed by condition (1.4). These orthogonal polynomials satisfy the three-term recurrence relation

$$x P_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n^2 P_{n-1}(x). \tag{1.6}$$

We will explore the asymptotics of the quantities γ_n, β_n, and h_n for $n = N, N-1$, as well as pointwise asymptotics of the polynomials $P_N(x)$ as $N \to \infty$.

The present work has the three predecessors:

(1) the work [6] of Deift, Kriecherbauer, McLaughlin, Venakides, and Zhou, in which the large N asymptotics has been obtained for orthogonal polynomials with respect to varying exponential weights on the real line,

(2) the work [1] of Baik, Kriecherbauer, McLaughlin, and Miller, in which the large N asymptotics has been obtained for orthogonal polynomials with respect to varying exponential weights on a lattice in a finite interval, and

(3) the work [3] of Bleher and Liechty, in which the large N asymptotics has been obtained for orthogonal polynomials with respect to the varying exponential weight $w_N(x) = e^{-N(|x|-\zeta x)}$ on the infinite lattice L_N.

Also, a very important ingredient comes from the work [11] of Kuijlaars, in which analytic properties of equilibrium measures with constraints are established.

The asymptotic analysis of the polynomials $P_N(x)$ in this work will be based on the Interpolation Problem for discrete orthogonal polynomials, which is introduced in the work [4] of Borodin and Boyarchenko (see also [1], [2], [3]). The asymptotic analysis of $P_N(x)$ will consist of three steps. The first step will be a reduction of the Interpolation Problem to a Riemann-Hilbert problem on a contour on the complex plane, which we accomplish following the general approach introduced in the paper [12] of Miller and in the monograph [10] of Kamvissis, McLaughlin, and Miller. The second step will be an application of the nonlinear steepest descent method of Deift and Zhou [8] to the Riemann-Hilbert problem under consideration, and the third and final step will be a derivation of the asymptotic formulae both for the orthogonal polynomials $P_N(x)$ and for the recurrence coefficients. To apply the nonlinear steepest descent method to the orthogonal polynomials $P_N(x)$ we need to study the corresponding equilibrium measure.

2. Equilibrium Measure

The significance of the equilibrium measure is that, as we will see, it gives the limiting distribution of zeros of the polynomial $P_N(x)$. By definition, the equilibrium measure is a solution to a variational problem. Namely, let us consider the following set of probability measures on $\mathbb{R}^1$:

$$\mathcal{M} = \{0 \leq \nu \leq \sigma, \ \nu(\mathbb{R}^1) = 1\}, \tag{2.1}$$

where σ is the Lebesgue measure, and let us introduce the functional

$$H(\nu) = \iint \log \frac{1}{|x-y|} d\nu(x) d\nu(y) + \int V(x) d\nu(x), \qquad \nu \in \mathcal{M}. \tag{2.2}$$

The equilibrium measure minimizes this functional over some set of measures. In the case of continuous orthogonal polynomials, we minimize over the set of probability measures on the real line. However, in the case of discrete orthogonal polynomials, we must introduce the upper constraint, $\nu \leq \sigma$, in order to account for an interlacing property of the zeroes of orthogonal polynomials.

It is a general fact, (see, e.g. [15]) that for any system of polynomials orthogonal on the real line with respect to a real weight, the nth polynomial has n real distinct distinct zeroes. Furthermore, the zeroes of a system of discrete orthogonal polynomials satisfy an interlacing

property with regard to the location of the nodes of the lattice L_N, so that no more than one zero may lie between any pair of adjacent nodes. It therefore follows that, if we denote by μ_N the normalized counting measure on the zeroes of the Nth orthogonal polynomial in our system,

$$\mu_N(a,b) \leq b - a + \frac{1}{N} \quad \text{for any} \quad -\infty < a < b < \infty, \tag{2.3}$$

so that $\mu \leq \sigma$, where $\mu = \lim_{N \to \infty} \mu_N$. With this constraint in mind, we define

$$E_0 = \inf_{\nu \in \mathcal{M}} H(\nu). \tag{2.4}$$

It is possible to prove that there exists a unique minimizer ν_0, so that

$$E_0 = H(\nu_0), \tag{2.5}$$

see, e.g., the works of Saff and Totik [14], Dragnev and Saff [7] and Kuijlaars [11]. The minimizer is called the *equilibrium measure*.

The equilibrium measure ν_0 is uniquely determined by the *Euler-Lagrange variational conditions*: there exists a *Lagrange multiplier l* such that

$$2 \int \log|x - y| d\nu_0(y) - V(x) \begin{cases} \geq l & \text{for} \quad x \in \text{supp } \nu_0 \\ \leq l & \text{for} \quad x \in \text{supp } (\sigma - \nu_0), \end{cases} \tag{2.6}$$

see the works [5] of Deift and McLaughlin and [7]. In particular,

$$2 \int \log|x - y| d\nu_0(y) - V(x) = l \quad \text{for} \quad x \in \text{supp } \nu_0 \cap \text{supp } (\sigma - \nu_0). \tag{2.7}$$

The equilibrium measure ν_0 possesses a number of nice analytical properties, as shown by Kuijlaars in [11]. We will use these analytic properties, so let us discuss the results of [11].

First, observe that the constraint $\nu_0 \leq \sigma$ implies the existence of the density,

$$\rho(x) = \frac{d\nu_0}{dx}. \tag{2.8}$$

We can partition $\mathbb{R}$ into the three sets

$$I^0 = \{x \in \mathbb{R} : 2 \int \log|x - y| d\nu_0(y) - V(x) = l\}$$

$$I^+ = \{x \in \mathbb{R} : 2 \int \log|x - y| d\nu_0(y) - V(x) > l\} \tag{2.9}$$

$$I^- = \{x \in \mathbb{R} : 2 \int \log|x - y| d\nu_0(y) - V(x) < l\}.$$

The structure of the equilibrium measure is well described in the following theorem of Kuijlaars, obtained in [11].

Theorem 2.1. *(Kuijlaars) For any real analytic potential $V(x)$ satisfying (1.4), the following hold:*

(1) *The density $\rho(x)$ of the constrained equilibrium measure ν_0 (defined in (2.5)) is continuous.*
(2) *The sets I^+ and I^- are both finite unions of open intervals.*
(3) *The density ρ is real analytic on the open set $\{x : 0 < \rho(x) < 1\}$.*

(4) *The density ρ has the representation*

$$\rho(x) = \frac{1}{\pi}\sqrt{q_1^+(x)} \quad for \quad x \in I^0 \cup I^-, \tag{2.10}$$

where q_1^+ is the positive part of a function q_1 defined on $I^0 \cup I^-$ which is real analytic on the interior of $I^0 \cup I^-$. The function q_1 is negative on I^-, so that

$$\rho(x) = 0 \quad for \quad x \in I^-, \tag{2.11}$$

and it is nonnegative on I^0, so that

$$\rho(x) = \frac{1}{\pi}\sqrt{q_1(x)} \quad for \quad x \in I^0. \tag{2.12}$$

(5) *The density ρ has the representation*

$$\rho(x) = 1 - \frac{1}{\pi}\sqrt{q_2^+(x)} \quad for \quad x \in I^0 \cup I^+, \tag{2.13}$$

where q_2^+ is the positive part of a function q_2 defined on $I^0 \cup I^+$ which is real analytic on the interior of $I^0 \cup I^+$. The function q_2 is negative on I^+, so that

$$\rho(x) = 1 \quad for \quad x \in I^+, \tag{2.14}$$

and it is nonnegative on I_0, so that

$$\rho(x) = 1 - \frac{1}{\pi}\sqrt{q_2(x)} \quad for \quad x \in I^0. \tag{2.15}$$

Remark: It follows from equations (2.12) and (2.15) that

$$\frac{1}{\pi}\sqrt{q_1(x)} = 1 - \frac{1}{\pi}\sqrt{q_2(x)} \quad for \quad x \in I^0, \tag{2.16}$$

hence q_1 and q_2 uniquely determine each other.

Notice that, according to point (2) of this theorem, the connected components of I^0 are either closed intervals or isolated points. Since ν_0 has compact support, we can write

$$I^0 = \bigsqcup_{j=1}^{q}[\alpha_j, \beta_j] \tag{2.17}$$

where

$$\begin{aligned} \alpha_j \leq \beta_j \quad &for \quad j = 1, \ldots, q \\ \beta_j < \alpha_{j+1} \quad &for \quad j = 1, \ldots, q-1. \end{aligned} \tag{2.18}$$

Notice that the intervals $(-\infty, \alpha_1)$ and (β_q, ∞) are components of I^-. The interval (β_j, α_{j+1}) for $1 \leq j < q$ is a component of either I^+ or I^-. We therefore adopt the notation

$$\begin{aligned} \mathcal{A}_v &= \left\{ j \in \{1, \ldots, q-1\} : (\beta_j, \alpha_{j+1}) \subset I^- \right\} \\ \mathcal{A}_s &= \left\{ j \in \{1, \ldots, q-1\} : (\beta_j, \alpha_{j+1}) \subset I^+ \right\}. \end{aligned} \tag{2.19}$$

We will call an equilibrium measure ν_0 *regular* if the following hold:

(1) q_1 and q_2 are non-vanishing on the interior of I^0.
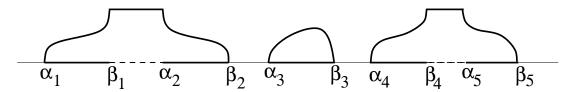(2) I^0 contains no isolated points, so that $\alpha_j < \beta_j$ for all $j = 1, \ldots, q$.

FIGURE 1. The graph of the density function for a hypothetical equilibrium measure with $q = 5$. Bands are denoted by bold segments, saturated regions by dashed segments, and voids by thin segments.

(3) If $j \in \mathcal{A}_v$, then $q_1'(\beta_j) \neq 0$ and $q_1'(\alpha_{j+1}) \neq 0$. Also, $q_1'(\alpha_1) \neq 0$ and $q_1'(\beta_q) \neq 0$.

(4) If $j \in \mathcal{A}_s$, then $q_2'(\beta_j) \neq 0$ and $q_2'(\alpha_{j+1}) \neq 0$.

For the remainder of this paper, we will assume that our equilibrium measure is regular. In this case, the sets I^0, I^+ and I^- are each finite unions of intervals, so that

$$-\infty < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_q < \beta_q < \infty, \tag{2.20}$$

and we classify these intervals as follows:

Definition: A *void* is an open subinterval (β_j, α_{j+1}), $j \in \mathcal{A}_v$, or one of the intervals $(-\infty, \alpha_1)$, (β_q, ∞). The union of all voids is I^-.

Definition: A *saturated region* is an open subinterval (β_j, α_{j+1}), $j \in \mathcal{A}_s$. The union of all saturated regions is I^+.

Definition: A *band* is an open subinterval (α_j, β_j), $j = 1, \ldots, q$. The union of all bands is the interior of I^0.

Observe that $\rho(x) = 0$ on any void (β_j, α_{j+1}), $\rho(x) = 1$ on any saturated interval (β_j, α_{j+1}), and $0 < \rho(x) < 1$ on any band (α_j, β_j), see Figure 1. In addition, at the end-points of any band, $\rho(x)$ has a square-root singularity. Namely, if α_j is a common end-point of a band and a void then as $x \to +0$,

$$\rho(\alpha_j + x) = C\sqrt{x}\,(1 + O(x)), \qquad C = |q_1'(\alpha_j)|^{1/2} > 0. \tag{2.21}$$

and if α_j is a common end-point of a band and a saturated region then as $x \to +0$,

$$\rho(\alpha_j + x) = 1 - C\sqrt{x}\,(1 + O(x)), \qquad C = |q_2'(\alpha_j)|^{1/2} > 0. \tag{2.22}$$

Similarly, if β_j is a common end-point of a band and a void then as $x \to +0$,

$$\rho(\beta_j - x) = C\sqrt{x}\,(1 + O(x)), \qquad C = |q_1'(\beta_j)|^{1/2} > 0. \tag{2.23}$$

and if β_j is a common end-point of a band and a saturated region then as $x \to +0$,

$$\rho(\beta_j - x) = 1 - C\sqrt{x}\,(1 + O(x)), \qquad C = |q_2'(\beta_j)|^{1/2} > 0. \tag{2.24}$$

In the next section we introduce the g-function, which will be our means of exploiting the equilibrium measure.

3. THE g-FUNCTION

Define the g-function on $\mathbb{C} \setminus (-\infty, \beta_q]$ as

$$g(z) = \int_{\alpha_1}^{\beta_q} \log(z-x)d\nu_0(x) \tag{3.1}$$

where we take the principal branch for the logarithm. Also, introduce the numbers Ω_j for $j = 1, \ldots, q-1$ as

$$\Omega_j = \begin{cases} 2\pi \displaystyle\int_{\alpha_{j+1}}^{\beta_q} \rho(x)dx & \text{for} \quad j \in \mathcal{A}_v \\ 2\pi \displaystyle\int_{\alpha_{j+1}}^{\beta_q} \rho(x)dx + 2\pi\alpha_{j+1} & \text{for} \quad j \in \mathcal{A}_s. \end{cases} \tag{3.2}$$

Properties of $g(z)$:
  (1) $g(z)$ is analytic in $\mathbb{C} \setminus (-\infty, \beta_q]$.
  (2) For large z,

$$g(z) = \log z - \sum_{j=1}^{\infty} \frac{g_j}{z^j}, \qquad g_j = \int_{\alpha_1}^{\beta_q} \frac{x^j}{j} \, d\nu_0(x). \tag{3.3}$$

  (3)

$$g'(z) = \int_{\mathbb{R}} \frac{\rho(x)dx}{z-x} \tag{3.4}$$

  is the resolvent of the equilibrium measure.
  (4) From (2.9), we have that

$$g_+(x) + g_-(x) \begin{cases} = V(x) + l & \text{for} \quad x \in I^0 \\ > V(x) + l & \text{for} \quad x \in I^+ \\ < V(x) + l & \text{for} \quad x \in I^-, \end{cases} \tag{3.5}$$

  where g_+ and g_- refer to the limiting values from the upper and lower half-planes, respectively.
  (5) Equation (3.1) implies that the function

$$G(x) \equiv g_+(x) - g_-(x) \tag{3.6}$$

  is pure imaginary for all real x, and

$$G(x) = 2\pi i \int_x^{\beta_q} \rho(s) \, ds. \tag{3.7}$$

  Thus

$$G(x) = \begin{cases} i\Omega_j & \text{for} \quad \beta_j < x < \alpha_{j+1}, \quad \text{and} \quad j \in \mathcal{A}_v \\ i\Omega_j - 2\pi i x & \text{for} \quad \beta_j < x < \alpha_{j+1}, \quad \text{and} \quad j \in \mathcal{A}_s. \end{cases} \tag{3.8}$$

  From (3.5) and (3.7) we obtain that

$$2g_\pm(x) = V(x) + l \pm 2\pi i \int_x^{\beta_q} \rho(s)ds \quad \text{for} \quad x \in I^0. \tag{3.9}$$

(6) Also, from (3.7), we get that $G(x)$ is real analytic on the sets I^+, I^-, and on the interior of I^0. We can therefore extend G into a complex neighborhood of any interval of analyticity for ρ, and the Cauchy-Riemann equations imply that

$$\left.\frac{dG(x+iy)}{dy}\right|_{y=0} = 2\pi\rho(x) \geq 0. \tag{3.10}$$

Observe that from (3.5) we have that

$$G(x) = 2g_+(x) - V(x) - l = -[2g_-(x) - V(x) - l], \quad x \in I^0. \tag{3.11}$$

4. Main Results

In this section, we summarize the main results of the paper. In order to do so, we must first introduce some notations. Introduce the numbers $\Omega_{j,N}$ for $j = 0, \ldots, q$ as

$$\Omega_{j,N} = \begin{cases} N\Omega_j & \text{for} \quad j \in \mathcal{A}_v \\ \pi + N\Omega_j & \text{for} \quad j \in \mathcal{A}_s \\ 2\pi N & \text{for} \quad j = 0 \\ 0 & \text{for} \quad j = q. \end{cases} \tag{4.1}$$

and the vector

$$\Omega_N = (\Omega_{1,N}, \ldots, \Omega_{q-1,N}). \tag{4.2}$$

Let

$$R(z) \equiv \prod_{j=1}^{q}(z - \alpha_j)(z - \beta_j) \tag{4.3}$$

and let X be the two-sheeted Riemann surface of genus $g \equiv q - 1$ associated with $\sqrt{R(z)}$ with cuts on the intervals (α_j, β_j). We fix the first sheet of X by the condition

$$\sqrt{R(z)} > 0 \quad \text{for} \quad z > \beta_q \tag{4.4}$$

on the first sheet.

Introduce the following homology basis on X. For any $j \in \{1, \cdots, q-1\}$, let A_j be a cycle enclosing the interval (β_j, α_{j+1}) (passing through the intervals (α_j, β_j) and $(\alpha_{j+1}, \beta_{j+1})$), oriented clockwise, such that the piece of A_j which lies in the upper half-plane also lies on the first sheet of X, while the piece of A_j which lies in the lower half plane also lies on the second sheet of X. Also for any $j \in \{1, \cdots, q-1\}$, let B_j be a cycle enclosing the interval (α_1, β_j) (passing through the intervals $(-\infty, \alpha_1)$ and (β_j, α_{j+1})), oriented clockwise, and lying entirely on the first sheet of X. Then the cycles $(A_1, \ldots, A_{q-1}, B_1, \ldots .B_{q-1})$ form a canonical homology basis for X.

Now consider the the g-dimensional complex linear space Ω of holomorphic one-forms on X,

$$\Omega = \left\{ \omega = \sum_{j=0}^{q-2} \frac{c_j z^j dz}{\sqrt{R(z)}} \right\}, \tag{4.5}$$

and the basis

$$\omega = (\omega_1, \ldots, \omega_{q-1}) \tag{4.6}$$

normalized such that

$$\int_{A_j} \omega_k = \delta_{jk}. \tag{4.7}$$

Notice that the basis ω is real. That is, for the basis elements

$$\omega_j = \sum_{k=1}^{q-1} \frac{c_{jk} z^{k-1} dz}{\sqrt{R(z)}}, \tag{4.8}$$

the coefficients c_{jk} are real.

Now define the associated matrix of B-periods as

$$\tau = (\tau_{jk}), \qquad \tau_{jk} = \int_{B_j} \omega_k, \qquad j, k = 1, \ldots, q-1. \tag{4.9}$$

Since $\sqrt{R(z)}$ is pure imaginary on the intervals (α_j, β_j), the numbers τ_{jk} are pure imaginary. Furthermore, the matrix τ is symmetric and the matrix $-i\tau$ is positive definite (see [9]).

We now define the Riemann theta function associated with τ as

$$\theta(s) = \sum_{m \in \mathbb{Z}^g} e^{2\pi i(m,s) + \pi i(m, \tau m)}, \qquad s \in \mathbb{C}^g, \tag{4.10}$$

where $(m, s) = \sum_{j=1}^{q-1} m_j s_j$. Because the quadratic form $i(m, \tau m)$ is negative definite, the sum in (4.10) is absolutely convergent for all $s \in \mathbb{C}^g$, and thus $\theta(s)$ is an entire function in $\mathbb{C}^g$. Notice that the theta function is an even function and satisfies the periodicity properties

$$\theta(s + e_j) = \theta(s), \qquad \theta(s + \tau_j) = e^{-2\pi i s_j - \pi i \tau_{jj}} \theta(s) \tag{4.11}$$

where $e_j = (0, \ldots, 1, \ldots, 0)$ is the jth canonical basis vector in $\mathbb{C}^g$, and $\tau_j = \tau e_j$.

Introduce now the vector valued function

$$u(z) = \int_{\beta_q}^{z} \omega, \quad \text{for} \quad z \in \mathbb{C} \setminus (\alpha_1, \beta_q), \tag{4.12}$$

where $\omega = (\omega_1, \ldots, \omega_g)$ is defined in (4.6), and the contour of integration lies in $\mathbb{C} \setminus (\alpha_1, \beta_q)$ on the first sheet of X. Notice that $u(z)$ is well defined as a function with values in $\mathbb{C}^g / \mathbb{Z}^g$ except on the interval (α_1, β_q), where it takes limiting values from the upper and lower half-planes.

Introduce also the function

$$\gamma(z) = \prod_{j=1}^{q} \left(\frac{z - \alpha_j}{z - \beta_j} \right)^{1/4} \tag{4.13}$$

with cuts on I^0, taking the branch such that $\gamma(z) \sim 1$ as $z \to \infty$. It can be seen that, on the first sheet of X, the function $\gamma - \frac{1}{\gamma}$ has exactly one zero in each of the intervals (β_j, α_{j+1}), and is non-zero elsewhere, and that the function $\gamma + \frac{1}{\gamma}$ has no zeroes on the first sheet of X. Define the numbers x_j as

$$x_j \in (\beta_j, \alpha_{j+1}), \quad \gamma(x_j) - \frac{1}{\gamma(x_j)} = 0. \tag{4.14}$$

Define the vector of Riemann constants

$$K \equiv - \sum_{j=1}^{q-1} u(\beta_j) \tag{4.15}$$

and the vector

$$d \equiv -K + \sum_{j=1}^{q-1} u(x_j). \tag{4.16}$$

Then

$$\theta(u(x_j) - d) = 0 \quad \text{for} \quad j \in \{1, \ldots, q-1\}, \tag{4.17}$$

and $\{x_j\}_{j=1}^{g}$ give all the zeroes of the function $\theta(u(z)-d)$. In addition, the function $\theta(u(z)+d)$ has no zeroes on the first sheet of X.

Finally, for $j = 1, \ldots q$, introduce the functions

$$\psi_{\alpha_j}(z) = -\left\{ \frac{3\pi}{2} \int_{\alpha_j}^{z} \rho(t)dt \right\}^{2/3}, \quad \psi_{\beta_j}(z) = -\left\{ \frac{3\pi}{2} \int_{z}^{\beta_j} \rho(t)dt \right\}^{2/3}, \tag{4.18}$$

and the functions

$$\begin{aligned}
\mathcal{M}_1(z) &= \frac{\theta(u(\infty) + d)}{\theta(u(\infty) + \frac{\Omega_N}{2\pi} + d)} \frac{\gamma(z) + \gamma(z)^{-1}}{2} \frac{\theta(u(z) + \frac{\Omega_N}{2\pi} + d)}{\theta(u(z) + d)} \\
\mathcal{M}_2(z) &= \frac{\theta(u(\infty) + d)}{\theta(u(\infty) + \frac{\Omega_N}{2\pi} + d)} \frac{\gamma(z) - \gamma(z)^{-1}}{2} \frac{\theta(u(z) - \frac{\Omega_N}{2\pi} - d)}{\theta(u(z) - d)}.
\end{aligned} \tag{4.19}$$

Notice that $\mathcal{M}_1$ and $\mathcal{M}_2$ depend quasiperiodically on N, thus are $O(1)$ as $N \to \infty$.

The asymptotics of the normalizing constants in equation (1.5) and of the recurrence coefficients in equation (1.6) are presented in the following theorem.

Theorem 4.1. *(Asymptotics of recurrence coefficients) Let $V(x)$ be a real analytic function satisfying (1.4) which yields a regular equilibrium measure (2.5), and let $\{P_n\}_{n=0}^{\infty}$ be the system of orthogonal polynomials defined according to (1.5). Then as $N \to \infty$, the normalizing constants in (1.5) and recurrence coefficients in (1.6) admit the following asymptotic expansions.*

$$h_N = \frac{N\pi}{2} e^{Nl} \left(\sum_{j=1}^{q} (\beta_j - \alpha_j) \right) \frac{\theta(u(\infty) + d)\theta(u(\infty) - \frac{\Omega_N}{2\pi} - d)}{\theta(u(\infty) - d)\theta(u(\infty) + \frac{\Omega_N}{2\pi} + d)} \left[1 + O\left(\frac{1}{N}\right) \right], \tag{4.20}$$

$$h_{N-1} = 8N\pi e^{Nl} \left(\sum_{j=1}^{q} (\beta_j - \alpha_j) \right)^{-1} \frac{\theta(u(\infty) - d)\theta(u(\infty) - \frac{\Omega_N}{2\pi} + d)}{\theta(u(\infty) + d)\theta(u(\infty) + \frac{\Omega_N}{2\pi} - d)} \left[1 + O\left(\frac{1}{N}\right) \right], \tag{4.21}$$

$$\gamma_N^2 = \left(\frac{\sum_{j=1}^{q} (\beta_j - \alpha_j)}{4} \right)^2 \frac{\theta(u(\infty) + d)^2 \theta(u(\infty) - \frac{\Omega_N}{2\pi} - d)\theta(u(\infty) + \frac{\Omega_N}{2\pi} - d)}{\theta(u(\infty) - d)^2 \theta(u(\infty) + \frac{\Omega_N}{2\pi} + d)\theta(u(\infty) - \frac{\Omega_N}{2\pi} + d)} + O\left(\frac{1}{N}\right), \tag{4.22}$$

$$\begin{aligned}
\beta_{N-1} &= \frac{\sum_{j=1}^{q} (\beta_j^2 - \alpha_j^2)}{2\sum_{j=1}^{q} (\beta_j - \alpha_j)} + \left(\frac{\nabla\theta(u(\infty) + \frac{\Omega_N}{2\pi} - d)}{\theta(u(\infty) + \frac{\Omega_N}{2\pi} - d)} - \frac{\nabla\theta(u(\infty) + \frac{\Omega_N}{2\pi} + d)}{\theta(u(\infty) + \frac{\Omega_N}{2\pi} + d)} \right. \\
&\quad \left. + \frac{\nabla\theta(u(\infty) + b)}{\theta(u(\infty) + b)} - \frac{\nabla\theta(u(\infty) - d)}{\theta(u(\infty) - d)}, u'(\infty) \right) + O\left(\frac{1}{N}\right).
\end{aligned} \tag{4.23}$$

where $\nabla \theta$ is the gradient of θ,

$$u'(\infty) = (c_{1,q-1}, c_{2,q-1}, \ldots, c_{q-1,q-1}), \tag{4.24}$$

and the numbers c_{jk} are defined in (4.8).

Notice that, up to the lattice scaling factor N in the normalizing coefficients, these asymptotics are similar to the results obtained in [6] for continuous orthogonal polynomials.

The remaining theorems in this section present pointwise asymptotics of the polynomials $P_N(z)$ in various regions of the real line and complex plane.

Theorem 4.2. *(Asymptotics of $P_N(z)$ in voids) Let $K \subset \mathbb{C}$ be a compact set on the complex plane such that K does not intersect with the support of the equilibrium measure ν_0. Then for any $z \in K$, we have that*

$$P_N(z) = e^{Ng(z)} \left[\mathcal{M}_1(z) + O(N^{-1}) \right]. \tag{4.25}$$

The error term $O(N^{-1})$ is uniform in K.

The function $e^{Ng(z)} \mathcal{M}_1(z)$ is analytic in a neighborhood of any compact subset of any void, thus this formula gives asymptotics of $P_N(x)$ for x in a void. In particular, notice that this function has no zeroes in the exterior intervals $(-\infty, \alpha_1)$ and (β_q, ∞), and at most one zero in any other void.

Theorem 4.3. *(Asymptotics of $P_N(z)$ in bands) Let K be a compact subset of the interior of I^0. Then for any point $x \in K$, we have that*

$$P_N(x) = 2e^{\frac{N}{2}(V(x)+l)} \left[\mathrm{Re} \left(e^{iN\pi\phi(x)} \mathcal{M}_{1+}(x) \right) + O(N^{-1}) \right], \tag{4.26}$$

where $\mathcal{M}_{1+}(x)$ refers to the limiting value of the function $\mathcal{M}_1(z)$ from the upper half-plane, and

$$\phi(x) := \int_x^{\beta_q} \rho(t)dt. \tag{4.27}$$

The error term $O(N^{-1})$ is uniform in K.

Theorem 4.4. *(Asymptotics of $P_N(z)$ in saturated regions) Let K be a compact subset of I^+. Then the exists $\varepsilon > 0$ such that, for any point $x \in K$, we have that*

$$P_N(x) = e^{NL(x)} \left[2\sin(N\pi x) \left(\mathrm{Im} \left(e^{\frac{iN\Omega_j}{2}} \mathcal{M}_{1+}(x) \right) + O(N^{-1}) \right) + O(e^{-N\varepsilon}) \right], \tag{4.28}$$

where $\mathcal{M}_{1+}(x)$ refers to the limiting value of the function $\mathcal{M}_1(z)$ from the upper half-plane, and

$$L(x) := \int_{\alpha_1}^{\beta_q} \log|x - t|\rho(t)dt. \tag{4.29}$$

Both of the error terms, $O(N^{-1})$ and $O(e^{-N\varepsilon})$, are uniform in K.

The remaining theorems in this section use the Airy functions Ai and Bi (see, e.g. [13]).

Theorem 4.5. *(Asymptotics of $P_N(z)$ at band-void edge points) Let $j \in \mathcal{A}_v \cup \{q\}$, so that the point β_j is the right endpoint of a band and the left endpoint of a void. Then there exists $\varepsilon > 0$ such that, for $|z - \beta_j| < \varepsilon$,*

$$P_N(z) = e^{\frac{N}{2}(V(z)+l)} \left\{ N^{1/6} \psi_{\beta_j}(z)^{1/4} \mathrm{Ai}(N^{2/3} \psi_{\beta_j}(z)) \left[e^{\pm \frac{i\Omega_{j,N}}{2}} \mathcal{M}_1(z) + e^{\mp \frac{i\Omega_{j,N}}{2}} \mathcal{M}_2(z) + O(N^{-1}) \right] \right.$$
$$\left. - N^{-1/6} \psi_{\beta_j}(z)^{-1/4} \mathrm{Ai}'(N^{2/3} \psi_{\beta_j}(z)) \left[e^{\pm \frac{i\Omega_{j,N}}{2}} \mathcal{M}_1(z) - e^{\mp \frac{i\Omega_{j,N}}{2}} \mathcal{M}_2(z) + O(N^{-1}) \right] \right\}$$
(4.30)

for $\pm \mathrm{Im}\, z > 0$.

Let $j \in \mathcal{A}_v \cup \{0\}$, so that the point α_{j+1} is the left endpoint of a band and the right endpoint of a void. There exists $\varepsilon > 0$ such that, for $|z - \alpha_{j+1}| < \varepsilon$,

$$P_N(z) = e^{\frac{N}{2}(V(z)+l)} \left\{ N^{1/6} \psi_{\alpha_{j+1}j}(z)^{1/4} \mathrm{Ai}(N^{2/3} \psi_{\alpha_{j+1}}(z)) \left[e^{\pm \frac{i\Omega_{j,N}}{2}} \mathcal{M}_1(z) - e^{\mp \frac{i\Omega_{j,N}}{2}} \mathcal{M}_2(z) + O(N^{-1}) \right] \right.$$
$$\left. - N^{-1/6} \psi_{\alpha_{j+1}}(z)^{-1/4} \mathrm{Ai}'(N^{2/3} \psi_{\alpha_{j+1}}(z)) \left[e^{\pm \frac{i\Omega_{j,N}}{2}} \mathcal{M}_1(z) + e^{\mp \frac{i\Omega_{j,N}}{2}} \mathcal{M}_2(z) + O(N^{-1}) \right] \right\}$$
(4.31)

for $\pm \mathrm{Im}\, z > 0$.

Theorem 4.6. *(Asymptotics of $P_N(z)$ at band-saturated region edge points) Let $j \in \mathcal{A}_s$. Then the point β_j is the right endpoint of a band and the left endpoint of a saturated region. There exists $\varepsilon > 0$ such that, for $|z - \beta_j| < \varepsilon$,*

$$P_N(z) = e^{\frac{N}{2}(V(z)+l)} \left\{ N^{1/6} \psi_{\beta_j}(z)^{1/4} \mathcal{B}_1(z) \left[-e^{\pm \frac{i\Omega_{j,N}}{2}} \mathcal{M}_1(z) + e^{\mp \frac{i\Omega_{j,N}}{2}} \mathcal{M}_2(z) + O(N^{-1}) \right] \right.$$
$$\left. - N^{-1/6} \psi_{\beta_j}(z)^{-1/4} \mathcal{B}_2(z) \left[e^{\pm \frac{i\Omega_{j,N}}{2}} \mathcal{M}_1(z) + e^{\mp \frac{i\Omega_{j,N}}{2}} \mathcal{M}_2(z) + O(N^{-1}) \right] \right\}$$
(4.32)

for $\pm \mathrm{Im}\, z > 0$, where

$$\begin{aligned} \mathcal{B}_1(z) &= \cos(N\pi z)\mathrm{Ai}(N^{2/3}\psi_{\beta_j}(z)) + \sin(N\pi z)\mathrm{Bi}(N^{2/3}\psi_{\beta_j}(z)), \\ \mathcal{B}_2(z) &= \cos(N\pi z)\mathrm{Ai}'(N^{2/3}\psi_{\beta_j}(z)) + \sin(N\pi z)\mathrm{Bi}'(N^{2/3}\psi_{\beta_j}(z)). \end{aligned}$$
(4.33)

The point α_{j+1} is the left endpoint of a band and the right endpoint of a void. There exists $\varepsilon > 0$ such that, for $|z - \alpha_{j+1}| < \varepsilon$,

$$P_N(z) = e^{\frac{N}{2}(V(z)+l)} \left\{ N^{1/6} \psi_{\alpha_{j+1}j}(z)^{1/4} \mathcal{B}_3(z) \left[e^{\pm \frac{i\Omega_{j,N}}{2}} \mathcal{M}_1(z) + e^{\mp \frac{i\Omega_{j,N}}{2}} \mathcal{M}_2(z) + O(N^{-1}) \right] \right.$$
$$\left. - N^{-1/6} \psi_{\alpha_{j+1}}(z)^{-1/4} \mathcal{B}_4(z) \left[e^{\pm \frac{i\Omega_{j,N}}{2}} \mathcal{M}_1(z) - e^{\mp \frac{i\Omega_{j,N}}{2}} \mathcal{M}_2(z) + O(N^{-1}) \right] \right\}$$
(4.34)

for $\pm \mathrm{Im}\, z > 0$, where

$$\begin{aligned} \mathcal{B}_3(z) &= \cos(N\pi z)\mathrm{Ai}(N^{2/3}\psi_{\alpha_{j+1}}(z)) - \sin(N\pi z)\mathrm{Bi}(N^{2/3}\psi_{\alpha_{j+1}}(z)), \\ \mathcal{B}_4(z) &= \cos(N\pi z)\mathrm{Ai}'(N^{2/3}\psi_{\alpha_{j+1}}(z)) - \sin(N\pi z)\mathrm{Bi}'(N^{2/3}\psi_{\alpha_{j+1}}(z)). \end{aligned}$$
(4.35)

Remark: Although the above theorems are presented for real analytic potential $V(x)$, these results may be extended to potentials which are continuous and piecewise real analytic, assuming that the points of non-analyticity lie strictly within saturated regions and voids. In

this case the preceding results hold, and the asymptotic solution to the associated Riemann-Hilbert Problem does not require local analysis near the points of non-analyticity (see [3]).

Before continuing with the proofs of these theorems, we would also like to remark that the results obtained in this paper match the results obtained in [1] for polynomials orthogonal on a lattice which sits inside a finite interval. Consequently, many corollaries discussed in [1] also apply to infinite lattices. In particular, the authors of [1] discuss the particle statistics of the *discrete orthogonal polynomial ensemble* in different regions of a finite interval of the real line, which are based on asymptotic properties of the associated orthogonal polynomials. The results of this paper imply that their results may be extended to discrete orthogonal polynomial ensembles on an infinite (regular) lattice. Of particular interest may be the *discrete sine kernel* as the scaling limit of the reproducing kernel in the interior of bands, the *Airy kernel* as the scaling limit of the reproducing kernel at band end-points, the *Tracy-Widom distribution* for the location of the left- and right-most particle, and exponential estimates for all correlation functions in voids and saturated regions.

The rest of the paper is organized as follows. In Section 5, we reformulate the orthogonal polynomials (1.4) as the solution to an interpolation problem of complex analysis. In Section 6, we reduce the interpolation problem to a Riemann-Hilbert Problem which can be solved by steepest descent analysis, which is done in Sections 7-12. Finally, in Section 13, we give proofs of the preceding theorems.

5. Interpolation Problem

We will evaluate the asymptotics of the discrete orthogonal polynomials described above via a steepest descent asymptotic analysis of a Riemann-Hilbert problem. To that end, we first introduce the following interpolation problem.

Interpolation Problem. For a given $N = 0, 1, \ldots$, find a 2×2 matrix-valued function $\mathbf{P}_N(z) = (\mathbf{P}_N(z)_{ij})_{1 \leq i,j \leq 2}$ with the following properties:

   (1) *Analyticity*: $\mathbf{P}_N(z)$ is an analytic function of z for $z \in \mathbb{C} \setminus L_N$.
   (2) *Residues at poles*: At each node $x \in L_N$, the elements $\mathbf{P}_N(z)_{11}$ and $\mathbf{P}_N(z)_{21}$ of the matrix $\mathbf{P}_N(z)$ are analytic functions of z, and the elements $\mathbf{P}_N(z)_{12}$ and $\mathbf{P}_N(z)_{22}$ have a simple pole with the residues,

$$\operatorname*{Res}_{z=x} \mathbf{P}_N(z)_{j2} = w_N(x)\mathbf{P}_N(x)_{j1}, \quad j = 1, 2. \tag{5.1}$$

   (3) *Asymptotics at infinity*: There exists a function $r(x) > 0$ on L_N such that

$$\lim_{x \to \infty} r(x) = 0, \tag{5.2}$$

and such that as $z \to \infty$, $\mathbf{P}_N(z)$ admits the asymptotic expansion,

$$\mathbf{P}_N(z) \sim \left(\mathbf{I} + \frac{\mathbf{P}_1}{z} + \frac{\mathbf{P}_2}{z^2} + \ldots\right)\begin{pmatrix} z^N & 0 \\ 0 & z^{-N} \end{pmatrix}, \qquad z \in \mathbb{C} \setminus \left[\bigcup_{x \in L_N}^{\infty} D(x, r(x))\right], \tag{5.3}$$

where $D(x, r(x))$ denotes a disk of radius $r(x) > 0$ centered at x and $\mathbf{I}$ is the identity matrix.

It is not difficult to see (see [4] and [1]) that the IP has a unique solution, which is

$$\mathbf{P}_N(z) = \begin{pmatrix} P_N(z) & C(w_N P_N)(z) \\ (h_{N-1})^{-1} P_{N-1}(z) & (h_{N-1})^{-1} C(w_N P_{N-1})(z) \end{pmatrix},$$
(5.4)

where the Cauchy transformation C is defined by the formula,

$$C(f)(z) = \sum_{x \in L_N} \frac{f(x)}{z - x}.$$
(5.5)

Because of the orthogonality condition, as $z \to \infty$,

$$C(w_N P_n)(z) = \sum_{x \in L_N} \frac{w_N(x) P_n(x)}{z - x} \sim \sum_{x \in L_N} w_N(x) P_n(x) \sum_{j=0}^{\infty} \frac{x^j}{z^{j+1}} = \frac{h_n}{z^{n+1}} + \sum_{j=n+2}^{\infty} \frac{a_j}{z^j},$$
(5.6)

which justifies asymptotic expansion (5.3), and have that

$$h_N = [\mathbf{P}_1]_{12}, \qquad h_{N-1}^{-1} = [\mathbf{P}_1]_{21}.$$
(5.7)

Furthermore, the recurrence coefficients in equation (1.6) are given by

$$\gamma_N^2 = [\mathbf{P}_1]_{12} [\mathbf{P}_1]_{21} \quad ; \quad \beta_{N-1} = \frac{[\mathbf{P}_2]_{21}}{[\mathbf{P}_1]_{21}} - [\mathbf{P}_1]_{11}.$$
(5.8)

6. REDUCTION OF IP TO RHP

We would like to reduce the Interpolation Problem to a Riemann-Hilbert Problem (RHP). Introduce the function

$$\Pi(z) = \frac{\sin N\pi z}{N\pi}.$$
(6.1)

Notice that

$$\Pi(x_k) = 0, \quad \Pi'(x_k) = \exp(iN\pi x_k) = (-1)^k, \quad \text{for} \quad x_k = \frac{k}{N} \in L_N.$$
(6.2)

Introduce the upper triangular matrices,

$$\mathbf{D}_{\pm}^u(z) = \begin{pmatrix} 1 & -\frac{w_N(z)}{\Pi(z)} e^{\pm iN\pi z} \\ 0 & 1 \end{pmatrix},$$
(6.3)

and the lower triangular matrices,

$$\mathbf{D}_{\pm}^l = \begin{pmatrix} \Pi(z)^{-1} & 0 \\ -\frac{1}{w_N(z)} e^{\pm iN\pi z} & \Pi(z) \end{pmatrix} = \begin{pmatrix} \Pi(z)^{-1} & 0 \\ 0 & \Pi(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{\Pi(z)w_N(z)} e^{\pm iN\pi z} & 1 \end{pmatrix}.$$
(6.4)

Define the matrix-valued functions,

$$\mathbf{R}_N^u = \mathbf{P}_N(z) \times \begin{cases} \mathbf{D}_+^u(z) & \text{when} \quad \text{Im}\, z \geq 0 \\ \mathbf{D}_-^u(z) & \text{when} \quad \text{Im}\, z \leq 0, \end{cases}$$
(6.5)

and

$$\mathbf{R}_N^l = \mathbf{P}_N(z) \times \begin{cases} \mathbf{D}_+^l(z), & \text{when} \quad \text{Im}\, z \geq 0 \\ \mathbf{D}_-^l(z), & \text{when} \quad \text{Im}\, z \leq 0. \end{cases}$$
(6.6)

From (5.4) we have that

$$\mathbf{R}_N^u(z) = \begin{pmatrix} P_N(z) & -\frac{w_N(z)P_N(z)}{\Pi(z)}e^{\pm iN\pi z} + C(w_N P_N)(z) \\ h_{N-1}^{-1}P_{N-1}(z) & -\frac{w_N(z)h_{N-1}^{-1}P_{N-1}(z)}{\Pi(z)}e^{\pm iN\pi z} + h_{N-1}^{-1}C(w_N P_{N-1})(z) \end{pmatrix} \qquad (6.7)$$

$$\text{when} \quad \pm \operatorname{Im} z \geq 0,$$

and

$$\mathbf{R}_N^l(z) = \begin{pmatrix} \frac{P_N(z)}{\Pi(z)} - \frac{C(w_N P_N)(z)}{w_N(z)}e^{\pm iN\pi z} & \Pi(z)C(w_N P_N)(z) \\ \frac{h_{N-1}^{-1}P_{N-1}(z)}{\Pi(z)} - \frac{h_{N-1}^{-1}C(w_N P_{N-1})(z)}{w_N(z)}e^{\pm iN\pi z} & \Pi(z)h_{N-1}^{-1}C(w_N P_{N-1})(z) \end{pmatrix} \qquad (6.8)$$

$$\text{when} \quad \pm \operatorname{Im} z \geq 0.$$

Observe that the functions $\mathbf{R}_N^u(z)$, $\mathbf{R}_N^l(z)$ are meromorphic on the closed upper and lower complex planes and they are two-valued on the real axis. Their possible poles are located on the lattice L_N. An important result is that, in fact, due to some cancellations they do not have any poles at all. We have the following proposition.

Proposition 6.1. *The matrix-valued functions $\mathbf{R}_N^u(z)$ and $\mathbf{R}_N^l(z)$ have no poles and on the real line they satisfy the following jump conditions at $x \in \mathbb{R}$:*

$$\mathbf{R}_{N+}^u(x) = \mathbf{R}_{N-}^u(x)j_R^u(x), \qquad j_R^u(x) = \begin{pmatrix} 1 & -2N\pi i w_N(x) \\ 0 & 1 \end{pmatrix}, \qquad (6.9)$$

and

$$\mathbf{R}_{N+}^l(x) = \mathbf{R}_{N-}^l(x)j_R^l(x), \qquad j_R^l(x) = \begin{pmatrix} 1 & 0 \\ -\frac{2N\pi i}{w_N(x)} & 1 \end{pmatrix}, \qquad (6.10)$$

Proof. It follows from the definition of $\mathbf{R}_N^u(z)$ that all possible poles of $\mathbf{R}_N^u(z)$ are located on the lattice L_N. Let us show that the residue of all these poles is equal to zero. Consider any $x_k \in L_N$. The residue of the matrix element $\mathbf{R}_{N,12}^u(z)$ at x_k is equal to

$$\operatorname*{Res}_{z=x_k} \mathbf{R}_{N,12}^u(z) = -\frac{w_N(x_k)P_N(x_k)}{(-1)^k}(-1)^k + w_N(x_k)P_N(x_k) = 0. \qquad (6.11)$$

Similarly we get that

$$\operatorname*{Res}_{z=x_k} \mathbf{R}_{N,22}(z) = 0, \qquad (6.12)$$

hence $\mathbf{R}_N^u(z)$ has no pole at x_k.

Similarly, the residue of the matrix element $\mathbf{R}_{N,11}^l(z)$ at x_k is equal to

$$\operatorname*{Res}_{z=x_k} \mathbf{R}_{N,11}^l(z) = \frac{P_N(x_k)}{(-1)^k} - \frac{w_N(x_k)P_N(x_k)(-1)^k}{w_N(x_k)} = 0. \qquad (6.13)$$

In the same way we obtain that

$$\operatorname*{Res}_{z=x_k} \mathbf{R}_{N,21}(z) = 0. \qquad (6.14)$$

In the entry $\mathbf{R}_{N,21}^l(z)$, the pole of the function $C(w_N P_N)(z)$ at $z = x_k$ is cancelled by the zero of the function $\Pi(z)$, hence $\mathbf{R}_{N,21}^l(z)$ has no pole at x_k. Similarly, $\mathbf{R}_{N,22}^l(z)$ has no pole at x_k as well, hence $\mathbf{R}_N^l(z)$ has no pole at x_k.
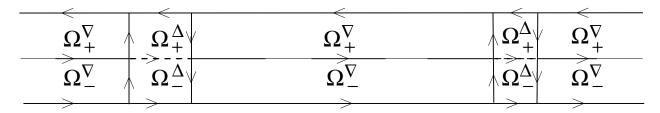
FIGURE 2. The contour Σ arising from the hypothetical equilibrium measure in Figure 1, dividing an ε-neighborhood of the real line into the regions $\Omega_\pm^\Delta$ and $\Omega_\pm^\nabla$.

Let us evaluate the jump matrices for $x \in \mathbb{R}$. From (6.5) we have that

$$j_R^u(x) = \mathbf{D}_-^u(x)^{-1}\mathbf{D}_+^u(x) = \begin{pmatrix} 1 & -\frac{w_N(x)}{\Pi(x)}2i\sin N\pi x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2N\pi i w_N(x) \\ 0 & 1 \end{pmatrix}, \qquad (6.15)$$

which proves (6.9). Similarly,

$$j_R^l(x) = \mathbf{D}_-^l(x)^{-1}\mathbf{D}_+^l(x) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{\Pi(x)w_N(x)}2i\sin N\pi x & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{2N\pi i}{w_N(x)} & 1 \end{pmatrix}, \qquad (6.16)$$

which proves (6.10). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

To reduce the Interpolation Problem to a Riemann-Hilbert Problem, we follow the work [1] with some modifications. Consider the oriented contour Σ on the complex plane depicted in Fig. 2, in which the horizontal lines are $\operatorname{Im} z = \varepsilon, 0, -\varepsilon$, where $\varepsilon > 0$ is a small positive constant which will be determined later, and the vertical segments pass through the endpoints of saturated intervals. Consider the regions

$$\begin{aligned} \Omega_\pm^\nabla &= \{I^0 \cup I^-\} \times (0, \pm i\varepsilon) \\ \Omega_\pm^\Delta &= I^+ \times (0, \pm i\varepsilon) \end{aligned} \qquad (6.17)$$

bounded by the contour Σ. Define

$$\mathbf{R}_N(z) = \begin{cases} \mathbf{K}_N \mathbf{R}_N^u(z)\mathbf{K}_N^{-1}, & \text{for} \quad z \in \Omega_\pm^\nabla, \\ \mathbf{K}_N \mathbf{R}_N^l(z)\mathbf{K}_N^{-1}, & \text{for} \quad z \in \Omega_\pm^\Delta, \\ \mathbf{K}_N \mathbf{P}_N(z)\mathbf{K}_N^{-1}, & \text{otherwise.} \end{cases} \qquad (6.18)$$

where $\mathbf{K}_N = \begin{pmatrix} 1 & 0 \\ 0 & -2iN\pi \end{pmatrix}$.

If $A \subset \mathbb{C}$ is a set on the complex plane and $b \in \mathbb{C}$ then, as usual, we denote

$$A + b = \{z = a + b, \ a \in A\}. \qquad (6.19)$$

Proposition 6.2. *The matrix-valued function $\mathbf{R}_N(z)$ has the following jumps on the contour Σ:*

$$\mathbf{R}_{N+}(z) = \mathbf{R}_{N-}(z)j_R(z), \qquad (6.20)$$

where

$$j_R(z) = \begin{cases} \begin{pmatrix} 1 & w_N(z) \\ 0 & 1 \end{pmatrix} & when \quad z \in I^- \cup I^0 \\[12pt] \begin{pmatrix} 1 & 0 \\ -(2N\pi)^2 w_N(z)^{-1} & 1 \end{pmatrix} & when \quad z \in I^+, \\[12pt] \mathbf{K}_N \mathbf{D}^u_\pm(z) \mathbf{K}_N^{-1} = \begin{pmatrix} 1 & \frac{1}{2iN\pi} \frac{w_N(z)e^{\pm iN\pi z}}{\Pi(z)} \\ 0 & 1 \end{pmatrix} & when \quad z \in \{I^- \cup I^0\} \pm i\varepsilon, \\[12pt] \mathbf{K}_N \mathbf{D}^l_\pm(z) \mathbf{K}_N^{-1} = \begin{pmatrix} \Pi(z)^{-1} & 0 \\ 2iN\pi \frac{e^{\pm iN\pi z}}{w_N(z)} & \Pi(z) \end{pmatrix} & when \quad z \in I^+ \pm i\varepsilon \\[12pt] \mathbf{K}_N \mathbf{D}^l_\pm(z)^{-1} \mathbf{D}^u_\pm(z) \mathbf{K}_N^{-1} = \begin{pmatrix} \Pi(z) & \frac{1}{2N\pi i} w_N(z) e^{\pm iN\pi z} \\ -2N\pi i w_N(z)^{-1} e^{\pm iN\pi z} & \mp 2N\pi i e^{\pm iN\pi z} \end{pmatrix} \\ \qquad when \quad z \in (0, \pm i\varepsilon) + \beta_j \quad or \quad z \in (0, \pm i\varepsilon) + \alpha_{j+1} \quad for \quad j \in \mathcal{A}_s. \end{cases}$$

$$(6.21)$$

7. First transformation of the RHP

Define the matrix function $\mathbf{T}_N(z)$ as follows from the equation,

$$\mathbf{R}_N(z) = e^{\frac{Nl}{2}\sigma_3} \mathbf{T}_N(z) e^{N(g(z) - \frac{l}{2})\sigma_3}, \qquad (7.1)$$

where l is the Lagrange multiplier, the function $g(z)$ is described in Section 2, and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the third Pauli matrix. Then $\mathbf{T}_N(z)$ satisfies the following Riemann-Hilbert Problem:

(1) $\mathbf{T}_N(z)$ is analytic in $\mathbb{C} \setminus \Sigma$.
(2) $\mathbf{T}_{N+}(z) = \mathbf{T}_{N-}(z) j_T(z)$ for $z \in \Sigma$, where

$$j_T(z) = \begin{cases} e^{N(g_-(z) - \frac{l}{2})\sigma_3} j_R(z) e^{-N(g_+(z) - \frac{l}{2})\sigma_3} & for \quad z \in \mathbb{R} \\ e^{N(g(z) - \frac{l}{2})\sigma_3} j_R(z) e^{-N(g(z) - \frac{l}{2})\sigma_3} & for \quad z \in \Sigma \setminus \mathbb{R}. \end{cases} \qquad (7.2)$$

(3) As $z \to \infty$,

$$\mathbf{T}_N(z) \sim \mathbf{I} + \frac{\mathbf{T}_1}{z} + \frac{\mathbf{T}_2}{z^2} + \dots. \qquad (7.3)$$

Let's take a closer look at the behavior of the jump matrix j_T described in (7.2) on the horizontal segments of Σ. We have that

$$
j_T(z) = \begin{cases}
\begin{pmatrix} e^{-NG(z)} & w_N(z)e^{N(g_+(z)+g_-(z)-l)} \\ 0 & e^{NG(z)} \end{pmatrix} & \text{when} \quad z \in I^0 \cup I^- \\[12pt]
\begin{pmatrix} e^{-NG(z)} & 0 \\ -(2N\pi)^2 e^{-N(g_+(z)+g_-(z)-V(z)-l)} & e^{NG(z)} \end{pmatrix} & \text{when} \quad z \in I^+ \\[12pt]
\begin{pmatrix} 1 & \pm\frac{e^{N(2g(z)-l-V(z))}}{1-e^{\mp 2iN\pi x}e^{\varepsilon 2N\pi}} \\ 0 & 1 \end{pmatrix} & \text{when} \quad z = x \pm i\varepsilon \in \{I^- \pm i\varepsilon\} \\[12pt]
\begin{pmatrix} 1 & \pm\frac{e^{\pm NG(z)}}{1-e^{\mp 2iN\pi x}e^{\varepsilon 2N\pi}} \\ 0 & 1 \end{pmatrix} & \text{when} \quad z = x \pm i\varepsilon \in \{I^0 \pm i\varepsilon\} \\[12pt]
\begin{pmatrix} \Pi(z)^{-1} & 0 \\ 2iN\pi e^{\pm iN\pi x}e^{-N(2g(z)-l-V(z))} & \Pi(z) \end{pmatrix} & \text{when} \quad z = x \pm i\varepsilon \in \{I^+ \pm i\varepsilon\}.
\end{cases}
\tag{7.4}
$$

8. Second transformation of the RHP

The second transformation is based on two observations. The first is the well known "opening of the lenses" in a neighborhood of the unconstrained support of the equilibrium measure. Namely, notice that, for $x \in I^0$, the jump matrix $j_T(x)$ factorizes as

$$
j_T(x) = \begin{pmatrix} e^{-NG(x)} & 1 \\ 0 & e^{NG(x)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{NG(x)} & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ e^{-NG(x)} & 1 \end{pmatrix}
$$
$$
= j_-(x)j_M j_+(x),
\tag{8.1}
$$

which allows us to reduce the jump matrix j_T to the one j_M plus asymptotically small jumps on the lens boundaries. The second observation consists of two facts. Firstly, the jumps on the segments $I^+ \pm i\varepsilon$ behave, for large N, as $\pm e^{\pm iN\pi z}$. Secondly, note that, for $x \in I^+$, $G(x)$ is a linear function with slope $-2\pi i$. With these facts in mind, we make the second transformation of the RHP. Let

$$
\mathbf{S}_N(z) = \begin{cases}
\mathbf{T}_N(z)j_+(z)^{-1} & \text{for} \quad z \in I^0 \times (0, i\varepsilon) \\
\mathbf{T}_N(z)j_-(z) & \text{for} \quad z \in I^0 \times (0, -i\varepsilon) \\
\mathbf{T}_N(z)\mathbf{A}_+(z) & \text{for} \quad z \in I^+ \times (0, i\varepsilon) \\
\mathbf{T}_N(z)\mathbf{A}_-(z) & \text{for} \quad z \in I^+ \times (0, -i\varepsilon) \\
\mathbf{T}_N(z) & \text{otherwise.}
\end{cases}
$$

where $\mathbf{A}_+(z) = \begin{pmatrix} -\frac{1}{2N\pi i}e^{-iN\pi z} & 0 \\ 0 & -2N\pi i e^{iN\pi z} \end{pmatrix}$ and $\mathbf{A}_-(z) = \begin{pmatrix} \frac{1}{2N\pi i}e^{iN\pi z} & 0 \\ 0 & 2N\pi i e^{-iN\pi z} \end{pmatrix}$.

$$\tag{8.2}$$

This function satisfies a similar RHP to $\mathbf{T}$, but jumps now occur on a new contour, Σ_S, which is obtained from Σ by adding the segments $(\alpha_1 - i\varepsilon, \alpha_1 + i\varepsilon)$, $(\beta_q - i\varepsilon, \beta_q + i\varepsilon)$, $(\alpha_{j+1} - i\varepsilon, \alpha_{j+1} + i\varepsilon)$, $(\beta_j - i\varepsilon, \beta_j + i\varepsilon)$ for $j \in \mathcal{A}_v$, see Fig. 3.
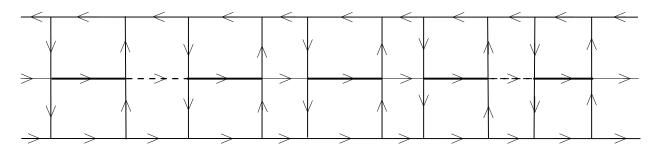
FIGURE 3. The contour Σ_S arising from the hypothetical equilibrium measure shown in Figure 1.

On horizontal segments, we have that

$$
j_S(z) = \begin{cases}
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for} \quad z \in I^0 \\[12pt]
\begin{pmatrix} 1 + O(e^{-2\varepsilon N\pi}) & O(e^{N(G(z)-2\varepsilon\pi)}) \\ -e^{-NG(z)} & 1 \end{pmatrix} & \text{for} \quad z - i\varepsilon \in I^0 \\[12pt]
\begin{pmatrix} 1 + O(e^{-2\varepsilon N\pi}) & O(e^{N(-G(z)-2\varepsilon\pi)}) \\ e^{NG(z)} & 1 \end{pmatrix} & \text{for} \quad z + i\varepsilon \in I^0 \\[12pt]
\begin{pmatrix} 1 + O(e^{-2\varepsilon N\pi}) & 0 \\ 2iN\pi e^{-N(2g(z)-l-V(z))} & 1 + O(e^{-2\varepsilon N\pi}) \end{pmatrix} & \text{for} \quad z \in \{I^+ \pm i\varepsilon\} \\[12pt]
\begin{pmatrix} -e^{-iN\Omega_j} & 0 \\ -e^{-N(g_+(z)+g_-(z)-l-V(z))} & -e^{iN\Omega_j} \end{pmatrix} & \text{for} \quad z \in (\beta_j, \alpha_{j+1}), \quad j \in \mathcal{A}_s \\[12pt]
\begin{pmatrix} 1 & e^{N(2g(z)-l-V(z))}O(e^{-2\varepsilon N\pi}) \\ 0 & 1 \end{pmatrix} & \text{for} \quad z = x \pm i\varepsilon \in \{I^- \pm i\varepsilon\} \\[12pt]
\begin{pmatrix} e^{-iN\Omega_j} & e^{N(g_+(z)+g_-(z)-l-V(z))} \\ 0 & e^{iN\Omega_j} \end{pmatrix} & \text{for} \quad z \in (\beta_j, \alpha_{j+1}), \quad j \in \mathcal{A}_v.
\end{cases}
$$
(8.3)

By formula (3.5) for the G-function and the upper constraint on the density ρ, we obtain that, for sufficiently small $\varepsilon > 0$ and $x \in (\alpha_j, \beta_j)$,

$$
0 < \mp \mathrm{Re}\, G(x \pm i\varepsilon) = 2\pi\rho(x) + O(\varepsilon^2) < 2\pi\varepsilon + O(\varepsilon^2).
$$
(8.4)

This, combined with property (3.7) of the g-function, implies that all jumps on horizontal segments are exponentially close to the identity matrix, provided that they are bounded away from the segment (α_1, β_q).

9. MODEL RHP

The model RHP appears when we drop in the jump matrix $j_S(z)$ the terms that vanish as $N \to \infty$:

(1) $\mathbf{M}(z)$ is analytic in $\mathbb{C} \setminus [\alpha_1, \beta_q]$.

(2) $\mathbf{M}_+(z) = \mathbf{M}_-(z)j_M(z)$ for $z \in [\alpha_1, \beta_q]$, where

$$j_M(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for} \quad z \in I^0 \\ e^{-i\Omega_{j,N}\sigma_3} & \text{for} \quad z \in (\beta_j, \alpha_{j+1}), \end{cases} \tag{9.1}$$

(3) As $z \to \infty$,

$$\mathbf{M}(z) \sim \mathbf{I} + \frac{\mathbf{M}_1}{z} + \frac{\mathbf{M}_2}{z^2} + \dots. \tag{9.2}$$

This model problem was first solved in the general multi-cut case in [6], and the solution is given as follows, using the notation introduced in section 4.

$$\mathbf{M}(z) = \mathbf{F}(\infty)^{-1} \begin{pmatrix} \frac{\gamma(z)+\gamma^{-1}(z)}{2} \frac{\theta(u(z)+\frac{\Omega_N}{2\pi}+d)}{\theta(u(z)+d)} & \frac{\gamma(z)-\gamma^{-1}(z)}{-2i} \frac{\theta(u(z)-\frac{\Omega_N}{2\pi}-d)}{\theta(u(z)-d)} \\ \frac{\gamma(z)-\gamma^{-1}(z)}{2i} \frac{\theta(u(z)+\frac{\Omega_N}{2\pi}-d)}{\theta(u(z)-d)} & \frac{\gamma(z)+\gamma^{-1}(z)}{2} \frac{\theta(u(z)-\frac{\Omega_N}{2\pi}+d)}{\theta(u(z)+d)} \end{pmatrix} \tag{9.3}$$

where

$$\mathbf{F}(\infty) = \begin{pmatrix} \frac{\theta(u(\infty)+\frac{\Omega_N}{2\pi}+d)}{\theta(u(\infty)+d)} & 0 \\ 0 & \frac{\theta(u(\infty)-\frac{\Omega_N}{2\pi}+d)}{\theta(u(\infty)+d)} \end{pmatrix}. \tag{9.4}$$

The asymptotics at infinity are given as

$$\mathbf{M}(z) = \mathbf{I} + \frac{\mathbf{M}_1}{z} + O(z^{-2}). \tag{9.5}$$

10. Parametrix at band-void edge points

We now consider small disks $D(\alpha_j, \varepsilon)$ for $j - 1 \in \mathcal{A}_v \cup \{0\}$, and $D(\beta_j, \varepsilon)$ for $j \in \mathcal{A}_v \cup \{q\}$, centered at the endpoints of bands which are adjacent to a void. Denote

$$D = \left(\bigcup_{j-1 \in \mathcal{A}_v \cup \{0\}} D(\alpha_j, \varepsilon) \right) \bigcup \left(\bigcup_{j \in \mathcal{A}_v \cup \{q\}} D(\beta_j, \varepsilon) \right). \tag{10.1}$$

We will seek a local parametrix $\mathbf{U}_N(z)$ defined on D such that

(1)
$$\mathbf{U}_N(z) \text{ is analytic on } D \setminus \Sigma_S. \tag{10.2}$$

(2)
$$\mathbf{U}_{N+}(z) = \mathbf{U}_{N-}(z)j_S(z) \quad \text{for} \quad z \in D \cap \Sigma_S. \tag{10.3}$$

(3)
$$\mathbf{U}_N(z) = \mathbf{M}(z)\big(I + O(N^{-1})\big) \quad \text{uniformly for } z \in \partial D. \tag{10.4}$$

We first construct the parametrix near β_j for $j \in \mathcal{A}_v$. The jumps j_S are given by

$$j_S(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in (\beta_j - \varepsilon, \beta_j) \\[2ex] \begin{pmatrix} 1 & 0 \\ -e^{-NG(z)} & 1 \end{pmatrix} & \text{for } z \in (\beta_j, \beta_j + i\varepsilon) \\[2ex] \begin{pmatrix} 1 & 0 \\ e^{NG(z)} & 1 \end{pmatrix} & \text{for } z \in (\beta_j, \beta_j - i\varepsilon) \\[2ex] \begin{pmatrix} e^{-NG(z)} & e^{N(g_+(z)+g_-(z)-V(z)-l)} \\ 0 & e^{NG(z)} \end{pmatrix} & \text{for } z \in (\beta_j, \beta_j + \varepsilon). \end{cases} \tag{10.5}$$

If we let

$$\mathbf{U}_N(z) = \mathbf{Q}_N(z) e^{-N(g(z) - \frac{V(z)}{2} - \frac{l}{2})\sigma_3}, \tag{10.6}$$

then the jump conditions on $\mathbf{Q}_N$ become

$$\mathbf{Q}_{N+}(z) = \mathbf{Q}_{N-}(z) j_Q(z) \tag{10.7}$$

where

$$j_Q(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in (\beta_j - \varepsilon, \beta_j) \\[2ex] \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & \text{for } z \in (\beta_j, \beta_j + i\varepsilon) \\[2ex] \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{for } z \in (\beta_j, \beta_j - i\varepsilon) \\[2ex] \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{for } z \in (\beta_j, \beta_j + \varepsilon). \end{cases} \tag{10.8}$$

where orientation is from left to right on horizontal contours, and down to up on vertical contours, according to Figure 3.

$\mathbf{Q}_N$ can be constructed using Airy functions. The Airy function solves the differential equation $y'' = zy$, and has the following asymptotics at infinity (see, e.g. [13]):

$$\begin{aligned} \mathrm{Ai}(z) &= \frac{1}{2\sqrt{\pi} z^{1/4}} e^{-\frac{2}{3} z^{3/2}} \left(1 - \frac{5}{48} z^{-3/2} + O(z^{-3}) \right) \\ \mathrm{Ai}'(z) &= -\frac{1}{2\sqrt{\pi}} z^{1/4} e^{-\frac{2}{3} z^{3/2}} \left(1 + \frac{7}{48} z^{-3/2} + O(z^{-3}) \right) \end{aligned} \tag{10.9}$$

as $z \to \infty$ with $\arg z \in (-\pi + \varepsilon, \pi - \varepsilon)$ for any $\varepsilon > 0$. If we let

$$y_0(z) = \mathrm{Ai}(z), \quad y_1(z) = \omega \mathrm{Ai}(\omega z), \quad y_2(z) = \omega^2 \mathrm{Ai}(\omega^2 z) \tag{10.10}$$

where $\omega = e^{\frac{2\pi i}{3}}$, then the functions y_0, y_1, and y_2 satisfy the relation

$$y_0(z) + y_1(z) + y_2(z) = 0. \tag{10.11}$$

If we take

$$
\Phi_{rv}(z) = \begin{cases}
\begin{pmatrix} y_0(z) & -y_2(z) \\ y_0'(z) & -y_2'(z) \end{pmatrix} & \text{for} \quad \arg z \in \left(0, \dfrac{\pi}{2}\right) \\[2.2em]
\begin{pmatrix} -y_1(z) & -y_2(z) \\ -y_1'(z) & -y_2'(z) \end{pmatrix} & \text{for} \quad \arg z \in \left(\dfrac{\pi}{2}, \pi\right) \\[2.2em]
\begin{pmatrix} -y_2(z) & y_1(z) \\ -y_2'(z) & y_1'(z) \end{pmatrix} & \text{for} \quad \arg z \in \left(-\pi, -\dfrac{\pi}{2}\right) \\[2.2em]
\begin{pmatrix} y_0(z) & y_1(z) \\ y_0'(z) & y_1'(z) \end{pmatrix} & \text{for} \quad \arg z \in \left(-\dfrac{\pi}{2}, 0\right),
\end{cases}
\tag{10.12}
$$

then Φ_{rv} satisfies jump conditions similar to (10.8), but for jumps on rays emanating from the origin rather than from β_j. We thus need to map the disk $D(\beta_j, \varepsilon)$ onto some convex neighborhood of the origin in order to take advantage of the function Φ_{rv}. Our mapping should match the asymptotics of the Airy function in order to have the matching property (10.4).

To this end, notice that, by (2.23), for $t \in [\alpha_j, \beta_j]$, as $t \to \beta_j$,

$$
\rho(t) = C(\beta_j - t)^{1/2} + O\big((\beta - t)^{3/2}\big), \quad C > 0.
\tag{10.13}
$$

It follows that, for $\quad x \in [\alpha_j, \beta_j] \quad$ as $\quad x \to \beta_j$,

$$
\int_x^{\beta_j} \rho(t)dt = C(\beta_j - x)^{3/2} + O\big((\beta_j - x)^{5/2}\big) \quad C_0 = \frac{2}{3}C.
\tag{10.14}
$$

Thus,

$$
\psi_{\beta_j}(z) = -\left\{\frac{3\pi}{2}\int_z^{\beta_j}\rho(t)dt\right\}^{2/3}
\tag{10.15}
$$

is analytic at β_j, thus extends to a conformal map from $D(\beta_j, \varepsilon)$ (for small enough ε) onto a convex neighborhood of the origin. Furthermore,

$$
\psi_{\beta_j}(\beta_j) = 0 \quad ; \quad \psi_{\beta_j}'(\beta_j) > 0,
\tag{10.16}
$$

thus ψ_{β_j} is real negative on $(\beta_j - \varepsilon, \beta_j)$, and real positive on $(\beta_j, \beta_j + \varepsilon)$. Also, we can slightly deform the vertical pieces of the contour Σ_S close to β_j, so that

$$
\psi_{\beta_j}\{D(\beta_j, \varepsilon) \cap \Sigma_S\} = (-\varepsilon, \varepsilon) \cup (-i\varepsilon, i\varepsilon).
\tag{10.17}
$$

We now set

$$
\mathbf{Q}_N(z) = \mathbf{E}_N^{\beta_j}(z)\Phi_{rv}\big(N^{2/3}\psi_{\beta_j}(z)\big)
\tag{10.18}
$$

so that

$$
\mathbf{U}_N(z) = \mathbf{E}_N^{\beta_j}(z)\Phi_{rv}\big(N^{2/3}\psi_{\beta_j}(z)\big)e^{-N(g(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3}
\tag{10.19}
$$

where

$$
\mathbf{E}_N^{\beta_j}(z) = \mathbf{M}(z)e^{\pm\frac{i\Omega_{j,N}}{2}\sigma_3}\mathbf{L}_N^{\beta_j}(z)^{-1} \quad \text{for} \quad \pm \operatorname{Im} z > 0,
$$

$$
\mathbf{L}_N^{\beta_j}(z) = \frac{1}{2\sqrt{\pi}}\begin{pmatrix} N^{-1/6}\psi_{\beta_j}^{-1/4}(z) & 0 \\ 0 & N^{1/6}\psi_{\beta_j}^{1/4}(z) \end{pmatrix}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix},
\tag{10.20}
$$

and we take the principal branch of $\psi_{\beta_j}^{1/4}$, which is positive on $(\beta_j, \beta_j + \varepsilon)$ and has a cut on $(\beta_j - \varepsilon, \beta_j)$. The function $\Phi_{rv}(N^{2/3}\psi_{\beta_j}(z))$ has the jumps j_S, and we claim that the prefactor $\mathbf{E}_N^{\beta_j}$ is analytic in $D(\beta_j, \varepsilon)$, thus does not change these jumps. This can be seen, as

$$\mathbf{M}_+(z)e^{\frac{i\Omega_{j,N}}{2}\sigma_3} = \mathbf{M}_-(z)e^{-\frac{i\Omega_{j,N}}{2}\sigma_3}e^{\frac{i\Omega_{j,N}}{2}\sigma_3}j_M e^{\frac{i\Omega_{j,N}}{2}\sigma_3} \tag{10.21}$$

thus the jump for the function $\mathbf{M}(z)e^{\pm\frac{i\Omega_{j,N}}{2}\sigma_3}$ is

$$e^{\frac{i\Omega_{j,N}}{2}\sigma_3}j_M e^{\frac{i\Omega_{j,N}}{2}\sigma_3} = \begin{cases} e^{\frac{i\Omega_{j,N}}{2}\sigma_3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}e^{\frac{i\Omega_{j,N}}{2}\sigma_3} & \text{for} \quad z \in (\beta_j - \varepsilon, \beta_j) \\ e^{\frac{i\Omega_{j,N}}{2}\sigma_3}e^{-i\Omega_{j,N}\sigma_3}e^{\frac{i\Omega_{j,N}}{2}\sigma_3} & \text{for} \quad z \in (\beta_j, \beta_j + \varepsilon) \end{cases} \tag{10.22}$$

or equivalently,

$$e^{\frac{i\Omega_{j,N}}{2}\sigma_3}j_M e^{\frac{i\Omega_{j,N}}{2}\sigma_3} = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for} \quad z \in (\beta_j - \varepsilon, \beta_j) \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for} \quad z \in (\beta_j, \beta_j + \varepsilon), \end{cases} \tag{10.23}$$

which is exactly the same as the jump conditions for $\mathbf{L}_N^{\beta_j}$. Thus $\mathbf{E}_N^{\beta_j}(z) = \mathbf{M}(z)e^{\pm\frac{i\Omega_{j,N}}{2}\sigma_3}\mathbf{L}_N^{\beta_j}(z)^{-1}$ has no jumps in $D(\beta_j, \varepsilon)$. The only other possible singularity for $\mathbf{E}_N^{\beta_j}$ is at β_j, and this singularity is at most a fourth root singularity, thus removable. Thus, $\mathbf{E}_N^{\beta_j}$ is analytic in $D(\beta_j, \varepsilon)$, and $\mathbf{Q}_N$ has the prescribed jumps. We are left only to prove the matching condition (10.4). Using (10.9), one can check that, for z in each of the sectors of analyticity, $\Phi_{rv}(N^{2/3}\psi_{\beta_j}(z))$ satisfies the following asymptotics as $N \to \infty$:

$$\Phi_{rv}\left(N^{2/3}\psi_{\beta_j}(z)\right) = \frac{1}{2\sqrt{\pi}}N^{-\frac{1}{6}\sigma_3}\psi_{\beta_j}(z)^{-\frac{1}{4}\sigma_3}\left[\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} + \frac{\psi_{\beta_j}(z)^{-3/2}}{48N}\begin{pmatrix} -5 & 5i \\ -7 & -7i \end{pmatrix} + O(N^{-2})\right]$$
$$\times e^{-\frac{2}{3}N\psi_{\beta_j}(z)^{3/2}\sigma_3} \tag{10.24}$$

where we always take the principal branch of $\psi_{\beta_j}(z)^{3/2}$. As such, $\psi_{\beta_j}(z)^{3/2}$ is two-valued for $z \in (\beta_j - \varepsilon, \beta_j)$, so that

$$\left[\frac{2}{3}\psi_{\beta_j}(x)^{3/2}\right]_\pm = \mp\pi i \int_x^{\beta_j} \rho(t)dt. \tag{10.25}$$

Notice that, by (3.9),

$$2g_\pm(x) - V(x) = l \pm 2\pi i \int_x^{\beta_q} \rho(t)dt = l \pm 2\pi i \int_x^{\beta_j} \rho(t)dt \pm i\Omega_j \tag{10.26}$$

This implies that for $x \in (\beta_j - \varepsilon, \beta_j)$,

$$[2g_+(\beta_j) - V(\beta_j)] - [2g_+(x) - V(x)] = -2\pi i \int_x^{\beta_j} \rho(t)dt,$$
$$[2g_-(\beta_j) - V(\beta_j)] - [2g_-(x) - V(x)] = 2\pi i \int_x^{\beta_j} \rho(t)dt. \tag{10.27}$$

Combining these equations with (10.25) gives

$$\left[\frac{2}{3}\psi_{\beta_j}(x)^{3/2}\right]_{\pm} = \frac{1}{2}\left[\left(2g_{\pm}(\beta_j) - V(\beta_j)\right) - \left(2g_{\pm}(x) - V(x)\right)\right].$$ (10.28)

This equation can be extended into the upper and lower planes, respectively, giving

$$\frac{2}{3}\psi_{\beta_j}(z)^{3/2} = \frac{1}{2}\left[\left(2g_{\pm}(\beta_j) - V(\beta_j)\right) - \left(2g(z) - V(z)\right)\right] \quad \text{for } \pm \operatorname{Im} z > 0.$$ (10.29)

Since, by (10.26), $2g_{\pm}(\beta_j) - V(\beta_j) = l \pm i\Omega_j$, we get that

$$\frac{2}{3}\psi_{\beta_j}(z)^{3/2} = -g(z) + \frac{V(z)}{2} + \frac{l}{2} \pm \frac{i\Omega_j}{2}$$ (10.30)

for $\pm\operatorname{Im} z > 0$. Plugging (10.24) and (10.30) into (10.19), we get

$$\mathbf{U}_N(z) = \mathbf{M}(z)e^{\pm\frac{i\Omega_{j,N}}{2}}\mathbf{L}_N^{\beta_j}(z)^{-1}\frac{1}{2\sqrt{\pi}}N^{-\frac{1}{6}\sigma_3}\psi_{\beta_j}(z)^{-\frac{1}{4}\sigma_3}\left[\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} + \frac{\psi_{\beta_j}(z)^{-3/2}}{48N}\begin{pmatrix} -5 & 5i \\ -7 & -7i \end{pmatrix}\right.$$

$$\left. + O(N^{-2})\right]e^{N(g(z)-\frac{V(z)}{2}-\frac{l}{2}\mp\frac{i\Omega_j}{2})\sigma_3}e^{-N(g(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3}$$

$$= \mathbf{M}(z)\left[\mathbf{I} + \frac{\psi_{\beta_j}(z)^{-3/2}}{48N}\begin{pmatrix} 1 & 6ie^{\pm i\Omega_{j,N}} \\ 6ie^{\mp i\Omega_{j,N}} & -1 \end{pmatrix} + O(N^{-2})\right]$$ (10.31)

for $\pm\operatorname{Im} z > 0$. Thus we have that $\mathbf{U}_N$ satisfies conditions (10.2), (10.3), and (10.4).

A similar construction gives the parametrix at the α_j for $j - 1 \in \mathcal{A}_v$. Namely, if we let

$$\psi_{\alpha_j}(z) = -\left\{\frac{3\pi}{2}\int_{\alpha_j}^z \rho(t)dt\right\}^{2/3},$$ (10.32)

then ψ_{α_j} is analytic throughout $D(\alpha_j, \varepsilon)$, real valued on the real line, and has negative derivative at α_j. Close to α_j, the jumps j_Q become

$$j_Q(z) = \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{for } z \in (\alpha_j - \varepsilon, \alpha_j) \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & \text{for } z \in (\alpha_j, \alpha_j + i\varepsilon) \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{for } z \in (\alpha_j, \alpha_j - i\varepsilon) \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in (\alpha_j, \alpha_j + \varepsilon), \end{cases}$$ (10.33)

where orientation is taken left to right on horizontal contours, and up to down on vertical contours according to Figure 3. After the change of variables ψ_{α_j} (and a slight deformation

of vertical contours), these jumps become the following jumps close to the origin:

$$
j_Q\big(\psi_{\alpha_j}(z)\big) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } \psi_{\alpha_j}(z) \in (-\varepsilon, 0) \\[2ex] \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{for } \psi_{\alpha_j}(z) \in (0, i\varepsilon) \\[2ex] \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & \text{for } \psi_{\alpha_j}(z) \in (0, -i\varepsilon) \\[2ex] \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{for } \psi_{\alpha_j}(z) \in (0, \varepsilon), \end{cases}
\tag{10.34}
$$

where orientation is taken right to left on horizontal contours, and down to up on vertical contours. These jump conditions are satisfied by the function

$$
\Phi_{lv}(z) = \Phi_{rv}(z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\tag{10.35}
$$

Then we can take

$$
\mathbf{U}_N(z) = \mathbf{E}_N^{\alpha_j}(z)\Phi_{lv}\big(N^{2/3}\psi_{\alpha_j}(z)\big)e^{-N(g(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3}
\tag{10.36}
$$

for $z \in D(\alpha_j, \varepsilon)$, where

$$
\mathbf{E}_N^{\alpha_j}(z) = \mathbf{M}(z)e^{\pm\frac{i\Omega_{j-1,N}}{2}\sigma_3}\mathbf{L}_N^{\alpha_j}(z)^{-1} \quad \text{for} \quad \pm\operatorname{Im} z > 0,
$$
$$
\mathbf{L}_N^{\alpha_j}(z) = \frac{1}{2\sqrt{\pi}} \begin{pmatrix} N^{-1/6}\psi_{\alpha_j}^{-1/4}(z) & 0 \\ 0 & N^{1/6}\psi_{\alpha_j}^{1/4}(z) \end{pmatrix} \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}
\tag{10.37}
$$

is an analytic prefactor. Similar to (10.24), we have that in each sector of analyticity, $\Phi_{lv}(N^{2/3}\psi_{\alpha_j}(z))$ satisfies

$$
\Phi_{lv}\big(N^{2/3}\psi_{\alpha_j}(z)\big) = \frac{1}{2\sqrt{\pi}}N^{-\frac{1}{6}\sigma_3}\psi_{\alpha_j}(z)^{-\frac{1}{4}\sigma_3}\left[\begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix} + \frac{\psi_{\alpha_j}(z)^{-3/2}}{48N}\begin{pmatrix} -5 & -5i \\ -7 & 7i \end{pmatrix} + O(N^{-2}) \right]
$$
$$
\times e^{-\frac{2}{3}N\psi_{\alpha_j}(z)^{3/2}\sigma_3}.
\tag{10.38}
$$

Once again, we have that, for $x \in (\alpha_j, \alpha_j + \varepsilon)$, $\psi_{\alpha_j}(x)^{3/2}$ takes limiting values from above and below, so that

$$
\left[\frac{2}{3}\psi_{\alpha_j}(x)^{3/2}\right]_\pm = \pm\pi i \int_{\alpha_j}^x \rho(t)dt.
\tag{10.39}
$$

In analogue to (10.28), we have

$$
\frac{2}{3}\psi_{\alpha_j}(z)^{3/2} = \frac{1}{2}\left[\big(2g_\pm(\alpha_j) - V(\alpha_j)\big) - \big(2g(z) - V(z)\big)\right] \quad \text{for} \quad \pm\operatorname{Im} z > 0.
\tag{10.40}
$$

Since, by (10.26), $2g_\pm(\alpha_j) - V(\alpha_j) = l \pm \pi i$, we get that

$$
\frac{2}{3}\psi_{\alpha_j}(z)^{3/2} = -g(z) + \frac{V(z)}{2} + \frac{l}{2} \pm \frac{i\Omega_{j-1}}{2} \quad \text{for} \quad \pm\operatorname{Im} z > 0.
\tag{10.41}
$$

Plugging (10.41) into (10.36) and (10.38) gives, as $N \to \infty$,

$$
\mathbf{U}_N(z) = \mathbf{M}(z)\mathbf{L}_N^{\alpha_j}(z)^{-1}\frac{1}{2\sqrt{\pi}}N^{-\frac{1}{6}\sigma_3}\psi_{\alpha_j}(z)^{-\frac{1}{4}\sigma_3}\left[\begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix} + \frac{\psi_{\alpha_j}(z)^{-3/2}}{48N}\begin{pmatrix} -5 & -5i \\ -7 & 7i \end{pmatrix}\right.
$$

$$
\left. + O(N^{-2})\right]e^{N(g(z)-\frac{V(z)}{2}-\frac{l}{2}\mp\frac{i\Omega_{j-1}}{2})\sigma_3}e^{-N(g(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3}
$$

$$
= \mathbf{M}(z)\left[\mathbf{I} + \frac{\psi_{\alpha_j}(z)^{-3/2}}{48N}\begin{pmatrix} 1 & -6ie^{\frac{i\Omega_{j-1,N}}{2}\sigma_3} \\ -6ie^{-\frac{i\Omega_{j-1,N}}{2}\sigma_3} & -1 \end{pmatrix} + O(N^{-2})\right].
$$

$$\tag{10.42}$$

11. PARAMETRIX AT THE BAND-SATURATED REGION END POINTS

We now consider small disks $D(\alpha_j, \varepsilon)$ for $j - 1 \in \mathcal{A}_s$, and $D(\beta_j, \varepsilon)$ for $j \in \mathcal{A}_s$, centered at the endpoints of bands which are adjacent to a saturated region. Denote

$$
\tilde{D} = \left(\bigcup_{j-1\in\mathcal{A}_s} D(\alpha_j, \varepsilon)\right)\bigcup\left(\bigcup_{j\in\mathcal{A}_s} D(\beta_j, \varepsilon)\right).
\tag{11.1}
$$

We will seek a local parametrix $\mathbf{U}_N(z)$ defined on $\tilde{D}$ such that

(1)
$$
\mathbf{U}_N(z) \text{ is analytic on } \tilde{D} \setminus \Sigma_S.
\tag{11.2}
$$

(2)
$$
\mathbf{U}_{N+}(z) = \mathbf{U}_{N-}(z)j_S(z) \quad \text{for} \quad z \in \tilde{D} \cap \Sigma_S.
\tag{11.3}
$$

(3)
$$
\mathbf{U}_N(z) = \mathbf{M}(z)\big(I + O(N^{-1})\big) \quad \text{uniformly for } z \in \partial\tilde{D}.
\tag{11.4}
$$

We first construct the parametrix near β_j for $j \in \mathcal{A}_s$. Let

$$
\mathbf{U}_N(z) = \tilde{\mathbf{Q}}_N(z)e^{\mp iN\pi z\sigma_3}e^{-N(g(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} \quad \text{for} \quad \pm\operatorname{Im} z > 0.
\tag{11.5}
$$

Then the jumps for $\tilde{\mathbf{Q}}_N$ are

$$
j_{\tilde{Q}}(z) = \begin{cases}
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for} \quad z \in (\beta_j - \varepsilon, \beta_j) \\[2mm]
\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} & \text{for} \quad z \in (\beta_j, \beta_j + \varepsilon) \\[2mm]
\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} & \text{for} \quad z \in (\beta_j, \beta_j + i\varepsilon) \\[2mm]
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{for} \quad z \in (\beta_j, \beta_j - i\varepsilon)
\end{cases}
\tag{11.6}
$$

where orientation is taken from left to right on horizontal contours, and down to up on vertical contours according to Figure 3. We now take

$$
\Phi_{rs}(z) = \begin{cases}
\begin{pmatrix} y_2(z) & -y_0(z) \\ y_2'(z) & -y_0'(z) \end{pmatrix} & \text{for} \quad \arg z \in \left(0, \frac{\pi}{2}\right) \\[2ex]
\begin{pmatrix} y_2(z) & y_1(z) \\ y_2'(z) & y_1'(z) \end{pmatrix} & \text{for} \quad \arg z \in \left(\frac{\pi}{2}, \pi\right) \\[2ex]
\begin{pmatrix} y_1(z) & -y_2(z) \\ y_1'(z) & -y_2'(z) \end{pmatrix} & \text{for} \quad \arg z \in \left(-\pi, -\frac{\pi}{2}\right) \\[2ex]
\begin{pmatrix} y_1(z) & y_0(z) \\ y_1'(z) & y_0'(z) \end{pmatrix} & \text{for} \quad \arg z \in \left(-\frac{\pi}{2}, 0\right).
\end{cases}
\tag{11.7}
$$

Then $\Phi_{rs}(z)$ solves a RHP similar to that of $\tilde{\mathbf{Q}}_N$, but for jumps emanating from the origin rather than from β_j.

Once again,

$$
\psi_{\beta_j}(z) = -\left\{ \frac{3\pi}{2} \int_z^{\beta_j} (1 - \rho(t))\, dt \right\}^{2/3}
\tag{11.8}
$$

extends to a conformal map from $D(\beta_j, \varepsilon)$ onto a convex neighborhood of the origin, with

$$
\psi_{\beta_j}(\beta_j) = 0 \quad ; \quad \psi_{\beta_j}'(\beta_j) > 0,
\tag{11.9}
$$

Again, we can slightly deform the vertical pieces of the contour Σ_S close to β_j, so that

$$
\psi_{\beta_j}\left\{ D(\beta_j, \varepsilon) \cap \Sigma_S \right\} = (-\varepsilon, \varepsilon) \cup (-i\varepsilon, i\varepsilon)
\tag{11.10}
$$

We now take

$$
\tilde{\mathbf{Q}}_N(z) = \mathbf{E}_N^{\beta_j}(z)\Phi_{rs}\left(N^{2/3}\psi_{\beta_j}(z)\right)
\tag{11.11}
$$

where

$$
\mathbf{E}_N^{\beta_j}(z) = \mathbf{M}(z) e^{\pm \frac{i\Omega_{j,N}}{2}\sigma_3} \mathbf{L}_N^{\beta_j}(z)^{-1} \quad \text{for} \quad \pm \operatorname{Im} z \geq 0,
$$

$$
\mathbf{L}_N^{\beta_j}(z) = \frac{1}{2\sqrt{\pi}} \begin{pmatrix} N^{-1/6}\psi_{\beta_j}^{-1/4}(z) & 0 \\ 0 & N^{1/6}\psi_{\beta_j}^{1/4}(z) \end{pmatrix} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix},
\tag{11.12}
$$

and we take the principal branch of $\psi_{\beta_j}^{1/4}$. The function $\Phi_{rs}(N^{2/3}\psi_{\beta_j}(z))$ has the jumps j_S. Similar to the prefactor $\mathbf{E}_N^{\beta_j}$ at band-void end-points, the prefactor $\mathbf{E}_N^{\beta_j}$ is analytic in $D(\beta_j, \varepsilon)$, thus does not change these jumps.

We now check that $\mathbf{U}_N$ satisfies the matching condition (11.4). The large N asymptotics of $\Phi_{rs}(N^{2/3}\psi_{\beta_j}(z))$ are given in the different regions of analyticity as follows:

$$
\Phi_{rs}\left(N^{2/3}\psi_{\beta_j}(z)\right) = \frac{1}{2\sqrt{\pi}} N^{-\frac{1}{6}\sigma_3}\psi_{\beta_j}(z)^{-\frac{1}{4}\sigma_3} \left[\pm \begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} \pm \frac{\psi_{\beta_j}(z)^{-3/2}}{48N} \begin{pmatrix} -5i & 5 \\ 7i & 7 \end{pmatrix} \right.
$$
$$
\left. + O(N^{-2}) \right] e^{\frac{2}{3}N\psi_{\beta_j}(z)^{3/2}\sigma_3} \quad \text{for} \quad \pm \operatorname{Im} z > 0,
\tag{11.13}
$$

where we always take the principal branch of $\psi_{\beta_j}(z)^{3/2}$. As such, $\psi_{\beta_j}(z)^{3/2}$ is two-valued for $x \in (\beta_j - \varepsilon, \beta_j)$, so that

$$\left[\frac{2}{3}\psi_{\beta_j}(x)^{3/2}\right]_\pm = \mp\pi i \int_x^{\beta_j} (1 - \rho(t))\, dt = \mp\pi i(\beta_j - x) \pm \pi i \int_x^{\beta_j} \rho(t)dt. \qquad (11.14)$$

From (3.9) we have that

$$2g_\pm(x) - V(x) = l \pm 2\pi i \int_x^{\beta_q} \rho(t)dt = l \pm 2\pi i \int_x^{\beta_j} \rho(t)dt \pm i\Omega_j \mp 2\pi i\beta_j \qquad (11.15)$$

for $x \in (\beta_j - \varepsilon, \beta_j)$. These equations imply that

$$\big(2g_\pm(x) - V(x)\big) - \big(2g_\pm(\beta_j) - V(\beta_j)\big) = \pm 2\pi i \int_x^{\beta_j} \rho(t)dt. \qquad (11.16)$$

We can therefore write (11.14) as

$$\left[\frac{2}{3}\psi_{\beta_j}(x)^{3/2}\right]_\pm = \mp\pi i(\beta_j - x) + \frac{1}{2}\left[\big(2g_\pm(x) - V(x)\big) - \big(2g_\pm(\beta_j) - V(\beta_j)\big)\right]. \qquad (11.17)$$

We can extend these equations into the upper and lower half-plane, respectively, obtaining

$$\frac{2}{3}\psi_{\beta_j}(z)^{3/2} = \mp\pi i(\beta_j - z) + \frac{1}{2}\left[\big(2g(z) - V(z)\big) - \big(2g_\pm(\beta_j) - V(\beta_j)\big)\right] \quad \text{for} \quad \pm\operatorname{Im} z > 0. \qquad (11.18)$$

Using (11.15) at $x = \beta_j$, we can write

$$\frac{2}{3}\psi_{\beta_j}(z)^{3/2} = g(z) - \frac{V(z)}{2} - \frac{l}{2} \pm \pi i z \mp \frac{i(\Omega_{j,N} - \pi)}{2N} \quad \text{for} \quad \pm\operatorname{Im} z > 0. \qquad (11.19)$$

Plugging (11.13) and (11.19) into (11.11) gives

$$\mathbf{U}_N(z) = \mathbf{M}(z)e^{\frac{i\Omega_{j,N}}{2}\sigma_3}\mathbf{L}_N^{\beta_j}(z)^{-1}\frac{1}{2\sqrt{\pi}}N^{-\frac{1}{6}\sigma_3}\psi_{\beta_j}(z)^{-\frac{1}{4}\sigma_3}$$

$$\times\left[\pm\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} \pm \frac{\psi_{\beta_j}(z)^{-3/2}}{48N}\begin{pmatrix} -5i & 5 \\ 7i & 7 \end{pmatrix} + O(N^{-2})\right]$$

$$\times e^{N(g(z)-\frac{l}{2}-\frac{V(z)}{2})\sigma_3}e^{\pm iN\pi z\sigma_3}e^{\mp\frac{i\Omega_{j,N}}{2}\sigma_3}e^{\pm\frac{i\pi}{2}\sigma_3}e^{\mp iN\pi z\sigma_3}e^{-N(g(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3}$$

$$= \mathbf{M}(z)\left[I + \frac{\psi_{\beta_j}(z)^{-3/2}}{48N}\begin{pmatrix} -1 & -6ie^{\pm i\Omega_{j,N}} \\ -6ie^{\mp i\Omega_{j,N}} & 1 \end{pmatrix} + O(N^{-2})\right] \quad \text{for} \quad \pm\operatorname{Im}(z) > 0.$$

$$(11.20)$$

We can make a similar construction near α_j for $j - 1 \in \mathcal{A}_s$. Let

$$\psi_{\alpha_j}(z) = -\left\{\frac{3\pi}{2}\int_{\alpha_j}^z (1 - \rho(t))\, dt\right\}^{2/3}. \qquad (11.21)$$

This function is analytic in $D(\alpha_j, \varepsilon)$ and has negative derivative at α_j, thus $\operatorname{Im} z$ and $\operatorname{Im} \psi_{\alpha_j}(z)$ have opposite signs for $z \in D(\alpha_j, \varepsilon)$. Then the jumps for $\tilde{\mathbf{Q}}_N$ are

$$j_{\tilde{Q}}(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for} \quad z \in (\alpha_j, \alpha_j + \varepsilon) \\[6pt] \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} & \text{for} \quad z \in (\alpha_j - \varepsilon, \alpha_j) \\[6pt] \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} & \text{for} \quad z \in (\alpha_j, \alpha_j + i\varepsilon) \\[6pt] \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{for} \quad z \in (\alpha_j, \alpha_j - i\varepsilon), \end{cases} \tag{11.22}$$

where the contour is oriented from left to right on horizontal segments and up to down on vertical segments according to Figure 3. After a slight deformation of the vertical contours and the change of variables ψ_{α_j}, these jumps become the following jumps close to the origin:

$$j_{\tilde{Q}}(\psi_{\alpha_j}(z)) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for} \quad \psi_{\alpha_j}(z) \in (-\varepsilon, 0) \\[6pt] \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} & \text{for} \quad \psi_{\alpha_j}(z) \in (0, \varepsilon) \\[6pt] \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} & \text{for} \quad \psi_{\alpha_j}(z) \in (-i\varepsilon, 0) \\[6pt] \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{for} \quad \psi_{\alpha_j}(z) \in (0, i\varepsilon), \end{cases} \tag{11.23}$$

where the contour is oriented from right to left on horizontal segments and down to up on vertical segments. These jump conditions are satisfied by the function

$$\Phi_{ls}(z) = \Phi_{rs}(z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{11.24}$$

Then we can take for $z \in D(\alpha_j, \varepsilon)$,

$$\mathbf{U}_N(z) = \mathbf{M}(z) e^{\frac{\pm i\Omega_{j,N}}{2}\sigma_3} \mathbf{L}_N^{\alpha_j}(z)^{-1} \Phi_{ls}(N^{2/3}\psi_{\alpha_j}(z)) e^{\mp iN\pi z \sigma_3} e^{-N(g(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} \quad \text{for} \quad \pm \operatorname{Im} z > 0, \tag{11.25}$$

where

$$\mathbf{L}_N^{\alpha_j}(z) = \frac{1}{2\sqrt{\pi}} \begin{pmatrix} N^{-1/6}\psi_{\alpha_j}^{-1/4}(z) & 0 \\ 0 & N^{1/6}\psi_{\alpha_j}^{1/4}(z) \end{pmatrix} \begin{pmatrix} -1 & i \\ -1 & -i \end{pmatrix}. \tag{11.26}$$

We once again have

$$\Phi_{ls}(N^{2/3}\psi_{\alpha_j}(z)) = \frac{1}{2\sqrt{\pi}} N^{-\frac{1}{6}\sigma_3} \psi_{\alpha_j}(z)^{-\frac{1}{4}\sigma_3} \left[\pm \begin{pmatrix} -i & 1 \\ -i & -1 \end{pmatrix} \pm \frac{\psi_{\alpha_j}(z)^{-3/2}}{48N} \begin{pmatrix} -5i & -5 \\ 7i & -7 \end{pmatrix} \right.$$
$$\left. + O(N^{-2}) \right] e^{\frac{2}{3}N\psi_{\alpha_j}(z)^{3/2}\sigma_3} \quad \text{for} \quad \pm \operatorname{Im} \psi_{\alpha_j}(z) > 0 \text{ (so } \mp \operatorname{Im} z > 0), \tag{11.27}$$

and for $z \in D(\alpha_j, \varepsilon)$,

$$\frac{2}{3}\psi_{\alpha_j}^{3/2}(z) = \pm i\pi z + g(z) - \frac{V(z)}{2} - \frac{l}{2} \mp \frac{i(\Omega_{j,N} - \pi)}{2N} \quad \text{for} \quad \pm \operatorname{Im} z > 0. \tag{11.28}$$

Combining (11.25), (11.27), and (11.28) gives

$$\mathbf{U}_N(z) = \mathbf{M}(z) e^{\frac{\pm i\Omega_{j,N}}{2}\sigma_3} \mathbf{L}_N^{\alpha_j}(z)^{-1} \frac{1}{2\sqrt{\pi}} N^{-\frac{1}{6}\sigma_3} \psi_{\alpha_j}(z)^{-\frac{1}{4}\sigma_3}$$

$$\times \left[\pm \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix} \pm \frac{\psi_{\alpha_j}(z)^{-3/2}}{48N} \begin{pmatrix} 5i & 5 \\ -7i & 7 \end{pmatrix} + O(N^{-2}) \right]$$

$$\times e^{\pm iN\pi z\sigma_3} e^{N(g(z) - \frac{V(z)}{2} - \frac{l}{2})\sigma_3} e^{\mp\frac{i\Omega_{j,N}}{2}\sigma_3} e^{\pm\frac{i\pi}{2}\sigma_3} e^{\mp iN\pi z\sigma_3} e^{-N(g(z) - \frac{V(z)}{2} - \frac{l}{2})\sigma_3}$$

$$= \mathbf{M}(z) e^{\frac{\pm i\Omega_{j,N}}{2}\sigma_3} \left[\mathbf{I} + \frac{\psi_{\alpha_j}(z)^{-3/2}}{48N} \begin{pmatrix} -1 & 6i \\ 6i & 1 \end{pmatrix} + O(N^{-2}) \right] e^{\mp\frac{i\Omega_{j,N}}{2}\sigma_3}$$

$$= \mathbf{M}(z) \left[\mathbf{I} + \frac{\psi_{\alpha_j}(z)^{-3/2}}{48N} \begin{pmatrix} -1 & 6ie^{\pm i\Omega_{j,N}} \\ 6ie^{\mp i\Omega_{j,N}} & 1 \end{pmatrix} + O(N^{-2}) \right] \quad \text{for} \quad \pm \operatorname{Im} z > 0. \tag{11.29}$$

12. The third and final transformation of the RHP

We now consider the contour Σ_X, which consists of the circles $\partial D(\alpha_j, \varepsilon)$, and $\partial D(\beta_j, \varepsilon)$, for $j = 1, \ldots q$, all oriented counterclockwise, together with the parts of $\Sigma_S \setminus \left(\bigcup_j [\alpha_j, \beta_j] \right)$ which lie outside of the disks $D(\alpha, \varepsilon)$, $D(\alpha', \varepsilon)$, $D(\beta', \varepsilon)$, and $D(\beta, \varepsilon)$, see Fig. 4.

FIGURE 4. The contour Σ_X arising from the hypothetical equilibrium measure shown in Figure 1.

We let

$$\mathbf{X}_N(z) = \begin{cases} \mathbf{S}_N(z)\mathbf{M}(z)^{-1} & \text{for } z \text{ outside the disks } D(\alpha_j, \varepsilon), \ D(\beta_j, \varepsilon) \\ \mathbf{S}_N(z)\mathbf{U}_N(z)^{-1} & \text{for } z \text{ inside the disks } D(\alpha_j, \varepsilon), \ D(\beta_j, \varepsilon). \end{cases} \tag{12.1}$$

Then $\mathbf{X}_N(z)$ solves the following RHP:
(1) $\mathbf{X}_N(z)$ is analytic on $\mathbb{C} \setminus \Sigma_X$.
(2) $\mathbf{X}_N(z)$ has the jump properties

$$\mathbf{X}_{N+}(x) = \mathbf{X}_{N-}(z) j_X(z) \tag{12.2}$$

where

$$j_X(z) = \begin{cases} \mathbf{M}(z)\mathbf{U}_N(z)^{-1} & \text{for } z \text{ on the circles} \\ \mathbf{M}(z)j_S\mathbf{M}(z)^{-1} & \text{otherwise.} \end{cases} \tag{12.3}$$

(3) As $z \to \infty$,

$$\mathbf{X}_N(z) \sim \mathbf{I} + \frac{\mathbf{X}_1}{z} + \frac{\mathbf{X}_2}{z^2} + \dots \tag{12.4}$$

Additionally, we have that $j_X(z)$ is uniformly close to the identity in the following sense:

$$j_X(z) = \begin{cases} \mathbf{I} + O(N^{-1}) & \text{uniformly on the circles} \\ \mathbf{I} + O(e^{-C(z)N}) & \text{on the rest of } \Sigma_X, \end{cases} \tag{12.5}$$

where $C(z)$ is a positive, continuous function satisfying (1.4). If we set

$$j_X^0(z) = j_X(z) - I, \tag{12.6}$$

then (12.5) becomes

$$j_X^0(z) = \begin{cases} O(N^{-1}) & \text{uniformly on the circles} \\ O(e^{-C(z)N}) & \text{on the rest of } \Sigma_X. \end{cases} \tag{12.7}$$

The solution to the RHP for $\mathbf{X}_N$ is based on the following lemma:

Lemma 12.1. *Suppose $v(z)$ is a function on Σ_X solving the equation*

$$v(z) = \mathbf{I} - \frac{1}{2\pi i} \int_{\Sigma_X} \frac{v(u)j_X^0(u)}{z_- - u} du \quad \text{for } z \in \Sigma_X \tag{12.8}$$

where z_- means the value of the integral on the minus side of Σ_X. Then

$$\mathbf{X}_N(z) = \mathbf{I} - \frac{1}{2\pi i} \int_{\Sigma_X} \frac{v(u)j_X^0(u)}{z - u} du \quad \text{for } z \in \mathbb{C} \setminus \Sigma_X \tag{12.9}$$

solves the RHP for $\mathbf{X}_N$.

The proof of this lemma is immediate from the jump property of the Cauchy transform. By assumption

$$\mathbf{X}_{N-}(z) = v(z) \tag{12.10}$$

and the additive jump of the Cauchy transform gives

$$\mathbf{X}_{N+}(z) - \mathbf{X}_{N-}(z) = v(z)j_X^0(z) = \mathbf{X}_{N-}(z)j_X^0(z) \tag{12.11}$$

thus $\mathbf{X}_{N+}(z) = \mathbf{X}_{N-}(z)j_X(z)$. Asymptotics at infinity are given by (12.9).

The solution to equation (12.8) is given by a series of perturbation theory. Namely, the solution is

$$v(z) = \mathbf{I} + \sum_{k=1}^{\infty} v_k(z) \tag{12.12}$$

where

$$v_k(z) = -\frac{1}{2\pi i} \int_{\Sigma_X} \frac{v_{k-1}(u)j_X^0(u)}{z - u} du \quad ; \quad v_0(z) = I. \tag{12.13}$$

This function clearly solves (12.8) provided the series converges, which it does, for sufficiently large N. Indeed, by (12.5),

$$|v_k(z)| \le \left(\frac{C}{N}\right)^k \frac{1}{1 + |z|} \quad \text{for some constant } C > 0, \tag{12.14}$$

thus the series (12.12) is dominated by a convergent geometric series and thus converges absolutely. This in turn gives

$$\mathbf{X}_N(z) = \mathbf{I} + \sum_{k=1}^{\infty} \mathbf{X}_{N,k}(z) \tag{12.15}$$

where

$$\mathbf{X}_{N,k}(z) = -\frac{1}{2\pi i} \int_{\Sigma_X} \frac{v_{k-1}(u) j_X^0(u)}{z - u} du. \tag{12.16}$$

In particular, this implies that

$$\mathbf{X}_N \sim \mathbf{I} + O\left(\frac{1}{N(|z|+1)}\right) \quad \text{as} \quad N \to \infty \tag{12.17}$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_X$.

13. Proof of theorems 4.1-4.6

The transformations (6.18), (7.1), (8.1), (12.1) give that, for z bounded away from the real line,

$$\mathbf{P}_N(z) = \mathbf{K}_N^{-1} e^{\frac{Nl}{2}\sigma_3} \mathbf{X}_N(z) \mathbf{M}(z) e^{N(g(z)-\frac{l}{2})\sigma_3} \mathbf{K}_N, \tag{13.1}$$

and for z close to the real line but bounded away from the support of the equilibrium measure,

$$\mathbf{P}_N(z) = \mathbf{K}_N^{-1} e^{\frac{Nl}{2}\sigma_3} \mathbf{X}_N(z) \mathbf{M}(z) e^{N(g(z)-\frac{l}{2})\sigma_3} \mathbf{K}_N \mathbf{D}_\pm^u(z)^{-1} \quad \text{for} \quad \pm \operatorname{Im} z \geq 0. \tag{13.2}$$

Expanding (13.1) or (13.2), we get that

$$P_N(z) = [\mathbf{P}_N(z)]_{11} = e^{Ng(z)} \left([\mathbf{M}]_{11}[\mathbf{X}]_{11} + [\mathbf{M}]_{21}[\mathbf{X}]_{12}\right) \tag{13.3}$$

which, along with (12.17), proves Theorem 4.2.

The proof of Theorem 4.1 requires only the formulae (5.3), (5.7), and (5.8), and a straightforward large-z expansion of equation (13.1).

Similar to (13.1), we have that, for any interval J which is contained in and bounded away from the endpoints of a band, in some neighborhood of J, we have

$$\mathbf{P}_N(z) = \begin{cases} \mathbf{K}_N^{-1} e^{\frac{Nl}{2}\sigma_3} \mathbf{X}_N(z) \mathbf{M}(z) j_+(z) e^{N(g(z)-\frac{l}{2})\sigma_3} \mathbf{K}_N \mathbf{D}_+^u(z)^{-1} & \text{for} \quad \operatorname{Im} z \geq 0 \\ \mathbf{K}_N^{-1} e^{\frac{Nl}{2}\sigma_3} \mathbf{X}_N(z) \mathbf{M}(z) j_-^{-1}(z) e^{N(g(z)-\frac{l}{2})\sigma_3} \mathbf{K}_N \mathbf{D}_-^u(z)^{-1} & \text{for} \quad \operatorname{Im} z \leq 0, \end{cases} \tag{13.4}$$

Expanding the left side of this equation for $\operatorname{Im} z \geq 0$, utilizing (3.9), and taking limits as z approaches the real line, we get that

$$P_N(x) = [\mathbf{P}_N(x)]_{11} = e^{\frac{N}{2}(V(x)+l)} \left(e^{iN\pi\phi(x)}[\mathbf{M}_{11}]_+(x) + e^{-iN\pi\phi(x)}[\mathbf{M}_{12}]_+(x) + O(N^{-1})\right) \tag{13.5}$$

where $\phi(x)$ is as defined in (4.27), and the $+$ subscript indicates the limiting value from the upper half plane. Notice that $[\mathbf{M}_{12}]_+ = [\mathbf{M}_{11}]_-$ in this region, and that $\mathbf{M}_{11}(\bar{z}) = \overline{\mathbf{M}_{11}(z)}$. This implies that $[\mathbf{M}_{12}]_+(x) = \overline{[\mathbf{M}_{11}]_+(x)}$, and thus we can write (13.5) as

$$P_N(x) = e^{\frac{N}{2}(V(x)+l)} \left(e^{iN\pi\phi(x)}[\mathbf{M}_{11}]_+(x) + \overline{e^{iN\pi\phi(x)}[\mathbf{M}_{11}]_+(x)} + O(N^{-1})\right), \tag{13.6}$$

which proves Theorem 4.3.

For any interval J which is contained in and bounded away from the endpoints of a saturated region, in some neighborhood of J, we have

$$\mathbf{P}_N(z) = \mathbf{K}_N^{-1} e^{\frac{Nl}{2}\sigma_3} \mathbf{X}_N(z) \mathbf{M}(z) \mathbf{A}_{\pm}^{-1}(z) e^{N(g(z)-\frac{l}{2})\sigma_3} \mathbf{K}_N \mathbf{D}_{\pm}^l(z)^{-1} \quad \text{for} \quad \pm \operatorname{Im} z > 0. \quad (13.7)$$

Notice that in this region, we can write

$$g_{\pm}(x) = L(x) \pm \frac{i\Omega_j}{2} \mp i\pi x \tag{13.8}$$

where $L(x)$ is defined in (4.29). Notice also that $2g_{\pm}(x) - V(x) - l$ has positive real part. Expanding (13.7) for $\operatorname{Im} z > 0$ and taking the limit as z approaches the real line gives

$$\begin{aligned}
\mathbf{P}_N(x)_{11} &= e^{Ng_+(x)} \left[(1 - e^{2\pi iNx})(\mathbf{M}_{11}\mathbf{X}_{11} + \mathbf{M}_{21}\mathbf{X}_{12}) + e^{-N(2g_+(x)-V(x)-l)}(\mathbf{M}_{12}\mathbf{X}_{11} + \mathbf{M}_{22}\mathbf{X}_{12}) \right] \\
&= e^{Ng_+(x)} \left[(1 - e^{2\pi iNx})(\mathbf{M}_{11}\mathbf{X}_{11} + O(N^{-1})) + O(e^{-N\delta}) \right] \\
&= e^{NL(x)} \left[-2i\sin(\pi Nx) e^{\frac{iN\Omega_j}{2}} [\mathbf{M}_{11}]_+(x)(1 + O(N^{-1})) + O(e^{-N\delta}) \right],
\end{aligned} \tag{13.9}$$

which proves Theorem 4.4.

Similarly, at the turning points α_j and β_j, explicit formulae can be written for $\mathbf{P}_N$ in terms of explicit transformations in each sector of analyticity of the local parametrix. From these formulae and the properties of the g-function, Theorems 4.5 and 4.6 are almost immediate, with Theorem 4.6 also requiring the identities (see, e.g. [13])

$$\begin{aligned}
y_1(z) &= -\frac{1}{2}\big(\operatorname{Ai}(z) - i\operatorname{Bi}(z)\big) \\
y_2(z) &= -\frac{1}{2}\big(\operatorname{Ai}(z) + i\operatorname{Bi}(z)\big).
\end{aligned} \tag{13.10}$$

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